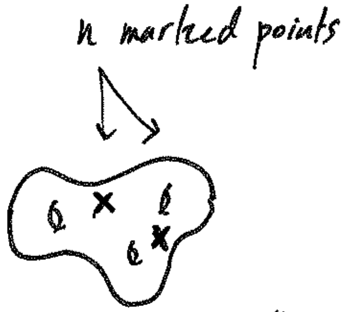
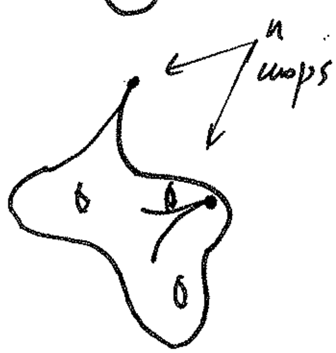
Bounds on lengths of simple closed geodesics on surfaces

Basic setup:

$\Sigma_{g,n}$: a topological orientable genus g surface with n punctures with $2g-2+n>0$



$\mathcal{T}_{g,n}$: associated Teichmüller space, i.e, set of marked complete finite area metrics one can induce $\Sigma_{g,n}$ with.



Interested in: basic things about the geometry of such surfaces like:

- diameter
- systole length / injectivity radius
- pants decompositions

and how these things relate.

<u>Basic example</u>: Bounds on the length of a <u>systole</u> of a surface.

<u>systole</u> = <u>systolic loop</u> = shortest closed geodesic of a surface.

= shortest simple closed geodesic of a surface

For $S \in \mathcal{T}_{g,n}$, denote the systole length $\mathrm{sys}(S)$.

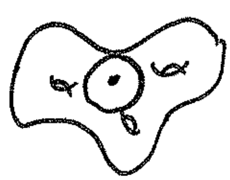
Consider <u>closed surfaces</u>, i.e., $n = 0$.

for $S \in \mathcal{T}_{g,0}$, $\mathrm{Area}(S) = 2\pi(2g-2)$

<u>In</u> $\mathbb{H}$:

$\mathbb{H}$

$\mathrm{Area}(\text{disk}) = \sinh r$



$\Rightarrow$ $\mathrm{Area}(\text{embedded disk}) < 2\pi(2g-2)$

$\Rightarrow$ non-trivial loop of length $\leq 2\log(g) + 10^{10}$

systole length is bounded :

$$\text{define} \quad sys(g) = \max_{S \in \mathcal{T}_{g,0}} sys(S)$$

Then :

$$10^{-10} \log(g) < sys(g) < 2 \log(g) + 10^{10}$$

just to be on the safe side .

In fact :

Theorem (Buser / Sarnak)
$\exists \{g_k\} \longrightarrow +\infty$ and S_k of genus g_k
with $sys(S_k)$ which behaves like $\frac{4}{3} \log g_k$

Questions :

1. Can one do better than 2 in front of the $\log g$?
 " " " " " $4/3$ " " " " $\log g_k$?

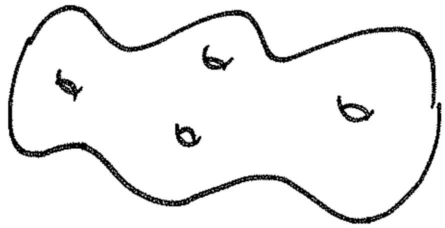
2. Is there assymptotic growth ?

3. Is $sys(g+1) > sys(g)$?

$$\boxed{50\$ \text{ conjecture : } \quad \lim_{g \to +\infty} \frac{sys(g)}{\log(g)} = 4/3}$$

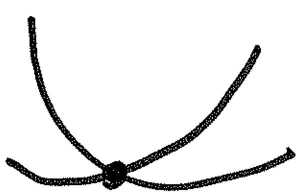
<u>Standard picture</u> :

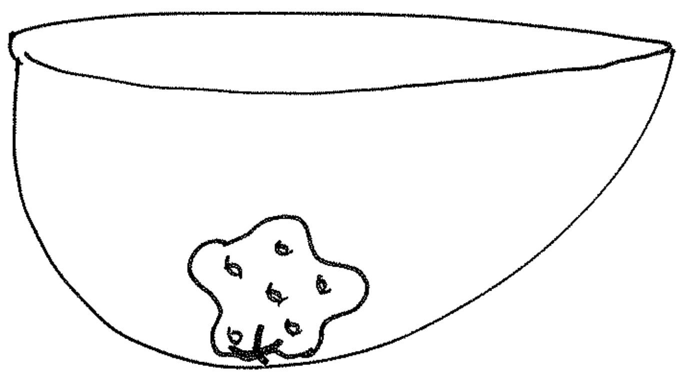
$(\mathbb{R}^3 \ldots)$



is wrong of course ...

<u>Salad bowl argument</u>: drop your surface in a salad bowl

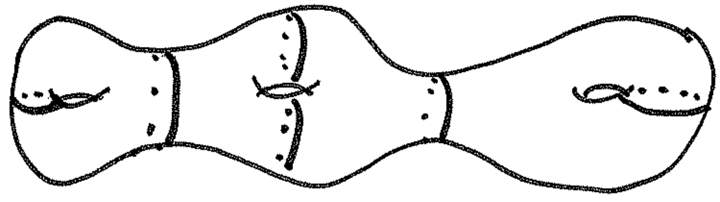




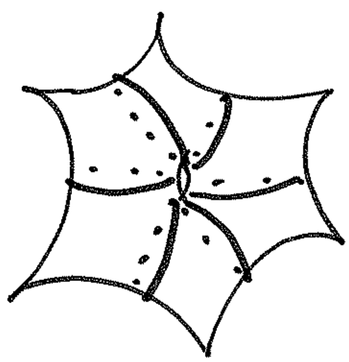
contact point has positive curvature

For $S \in \mathcal{T}_{g,n}$

consider a pants decomposition



$$\mathcal{P} = \{ \gamma_1 , \ldots , \gamma_{3g+3+n} \}$$



Define: $\ell(\mathcal{P}) = \max\limits_{k \in \{1,\ldots,3g-3+n\}} \ell(\gamma_k)$

Define: $\mathcal{B}(S) = \min\limits_{\mathcal{P} \text{ a p.d. of } S} \ell(\mathcal{P})$

Define: $\mathcal{B}_{g,n} := \sup\limits_{S \in \mathcal{T}_{g,n}} \mathcal{B}(S)$

Theorem (Bers) $B_{g,n} < +\infty$

$B_{g,n}$ are called Bers' constants.

Buser's theorem on Bers' constants:

Theorem: $\sqrt{6}\sqrt{g} - 2 < B_{g,0} < 19(g-1)$

$\vdots$

$B_{0,n} < 11(n-1)$

Interested in the rough growth

Remark: Buser's methods rely on area arguments like for the systole + induction to handle the combinatorics of pants decompositions.

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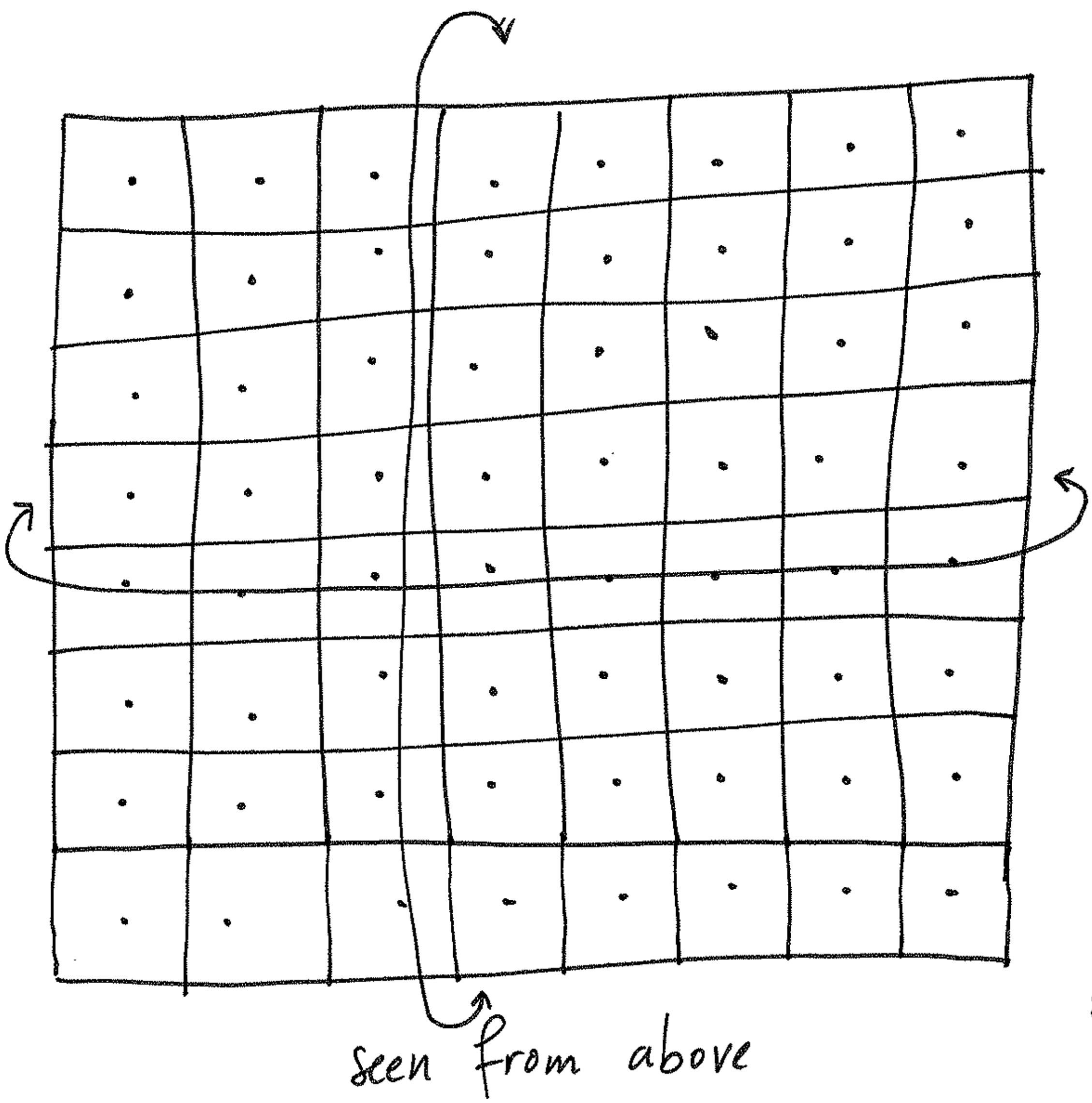
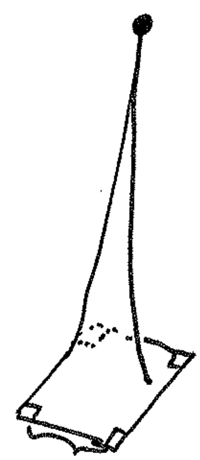
lower bound is based on the hairy torus:

$n = m^2$ cusps



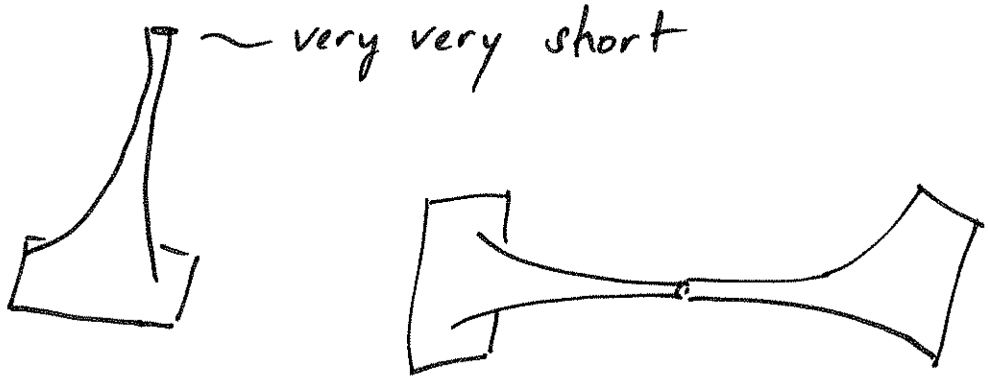
seen from above



2 arcsinh 1

shows $\mathcal{B}_{1, n=m^2} \geq 2 \operatorname{arcsinh} 1 \cdot \sqrt{n}$

adapted using collar lemma:



to get genus.

Buser's conjecture: $\exists\, c > 0$ s.t.

$$B_{g,n} < c\sqrt{g+n}$$

More generally one can conjecture:

In variable curvature:

$\exists\, K > 0$ s.t. any genus g surface has a p.d. of length at most $K\sqrt{\text{area of the surface}}$

Joint with F. Balacheff.

Theorem 1. $\mathcal{B}_{0,n} \leq 30\sqrt{2\pi(n-2)}$
(B-P)

Remark: By adapting the hairy torus argument we get a lower bound with rough growth $\sqrt{n}$.

Theorem 2. If S is hyperelliptic of genus g then $\mathcal{B}(S) < 24\sqrt{(2g-2)\,2\pi}$
(B-P)

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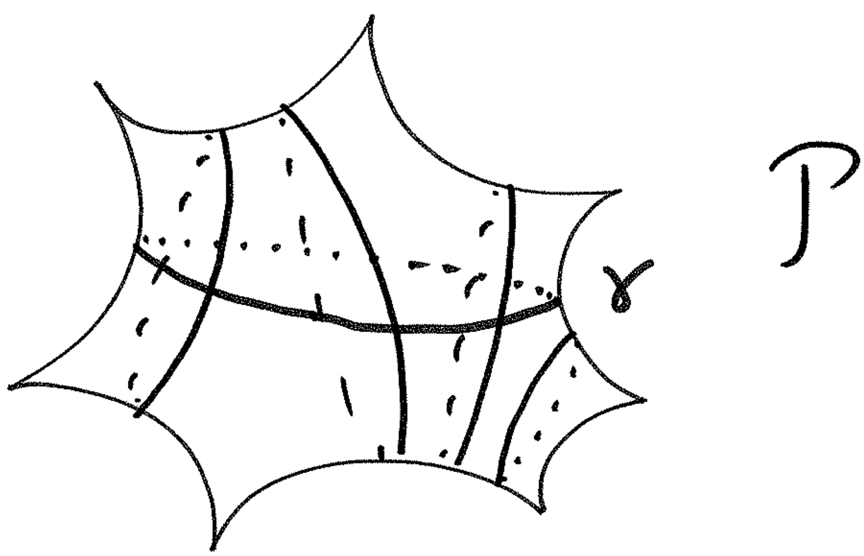
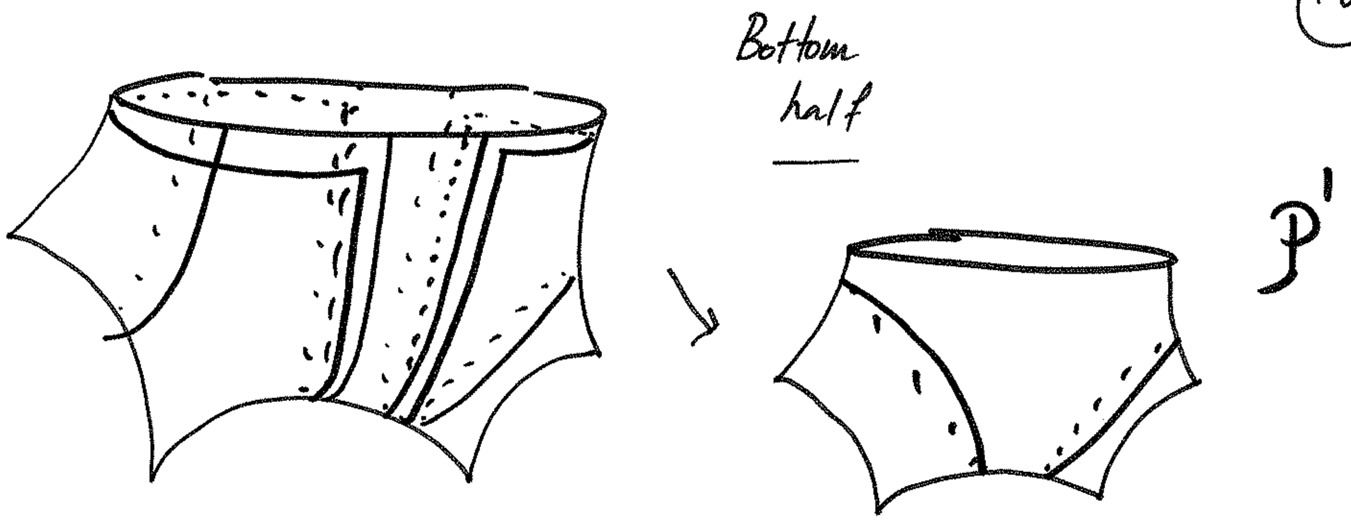
i.e. the conjecture holds for n - punctured spheres and hyperelliptic surfaces. Theorem 2 is really a byproduct of Theorem 1.

Idea of the proof of Theorem 1

Lemma 1: Consider $S \in T_{0,n}$, and $\mathcal{P}$ a pants decomposition of S. For any γ, simple closed geodesic of S, there exists a $\mathcal{P}'$, another pants decomposition, with $\gamma \subset \mathcal{P}'$ and $\ell(\mathcal{P}') \leq \ell(\gamma) + \ell(\mathcal{P})$.





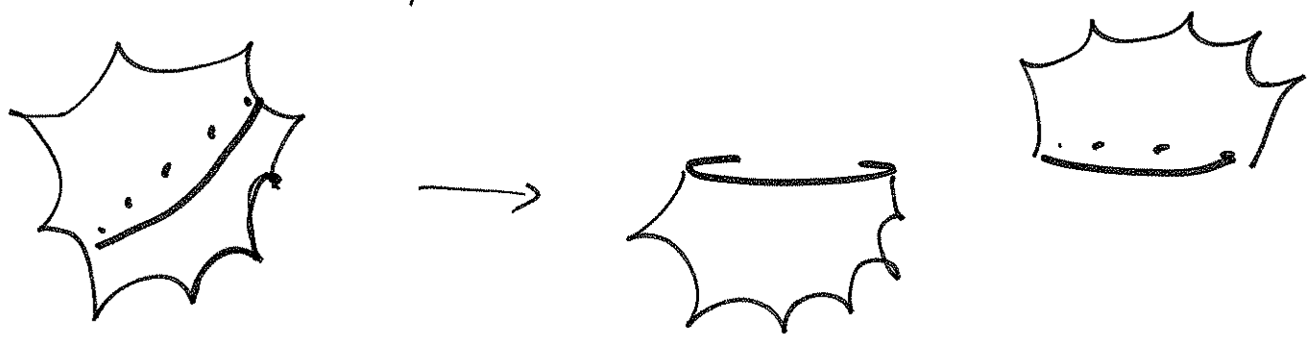
The proof is a generalization of the above picture.

Note that this process of projecting curves isn't uniquely defined, meaning that there is a choice involved!

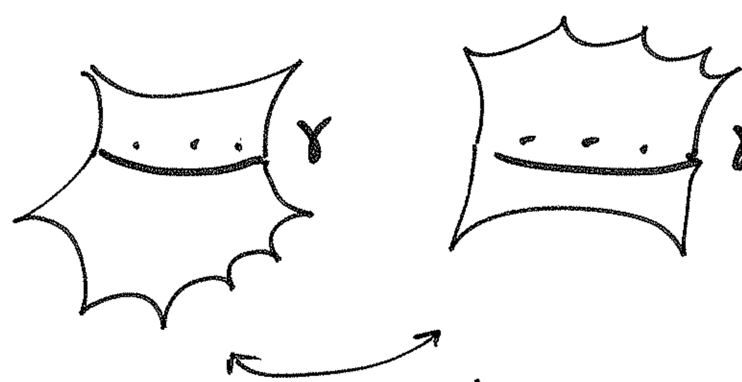
Philosophy of the rest of the proof :

→ argue by induction by doing the <u>following</u> :

- Find a short γ that cuts the surface into 2 parts (<u>both</u> big)



Add a <u>hat</u> + a <u>pair of pants</u> :



<u>less loops</u> !

thus shorter pants decompositions P_1 and P_2

use lemma 1

$\longrightarrow$ P_1' and P_2' containing γ $\rightarrow$ short P.

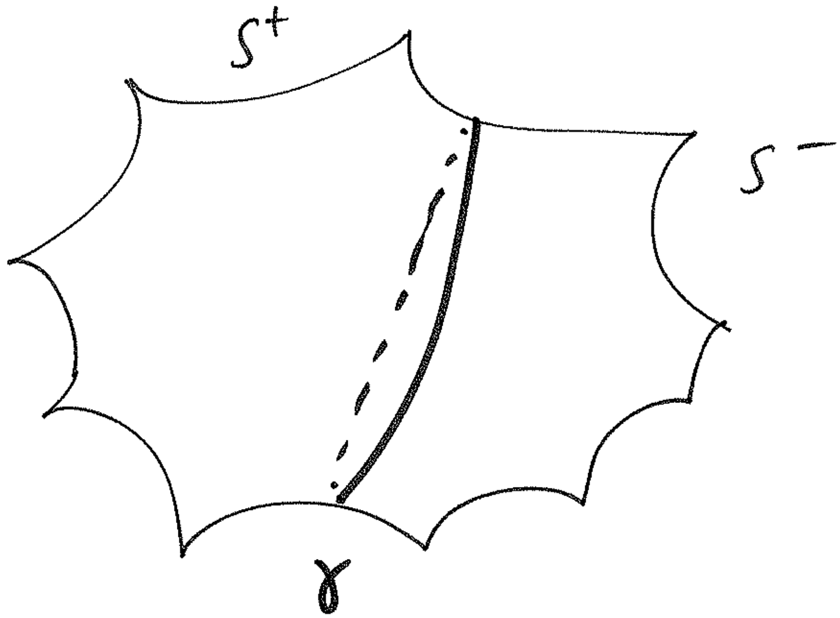
Problem: Finding a short γ.

Here's how to:

Lemma $\exists\ \gamma$ such that S^+, S^- both contain at least $n/4$ cusps

and

$$\ell(\gamma) \leq 4 \cdot \sqrt{2\pi(n/4 - 1)}$$



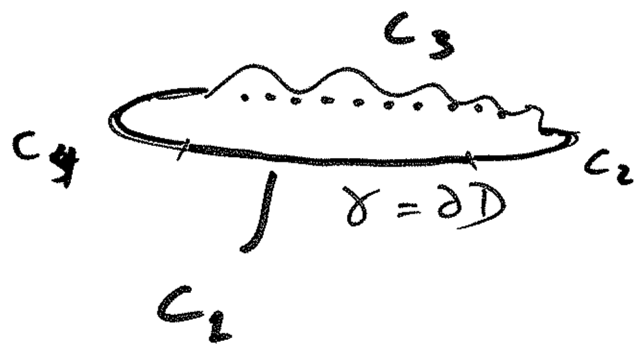
S^+ has n^+ cusps $\qquad n^+ \geq n^- \geq n/4$.

S^- has n^- cusps

proof: Uses the Besicovitch lemma (there exist different versions of this).

in dim 2.

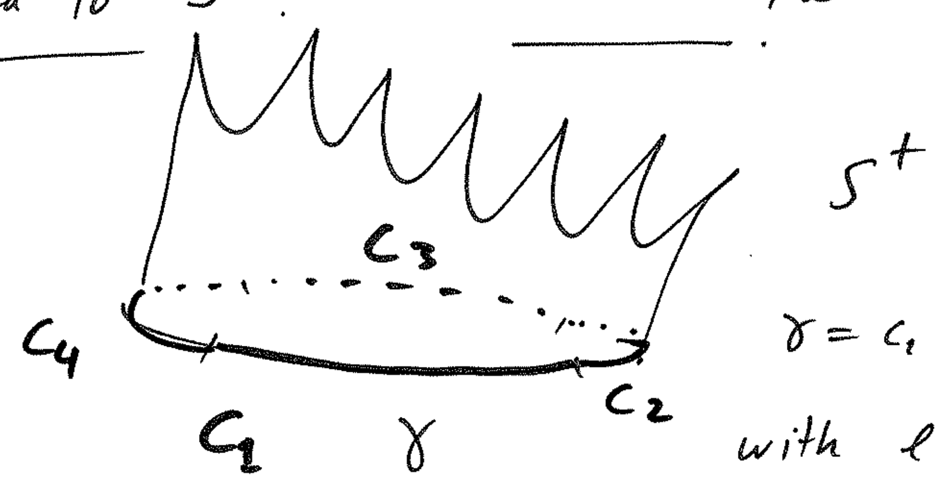
Besicovitch lemma: Let D be a riemannian disk and let $\gamma = \partial D$



Let $\gamma = c_1 + c_2 + c_3 + c_4$

then $\text{Area}(D) \geq d_D(c_1, c_3) \cdot d_D(c_2, c_4)$

Applied to S^+. Note $n^+ \geq n/2$.



$\gamma = c_1 + c_2 + c_3 + c_4$

with $\ell(c_k) = \dfrac{\ell(\gamma)}{4}$.

Then : $d_{S^+}(c_1, c_3) \geq \frac{\ell(\gamma)}{4}$

or there is a shorter γ.

$\ell(\gamma_2), \ell(\gamma_2) < \ell(\gamma)$

and one of them satisfies the cusp condition.

Thus : $\text{Area}(S^+) \geq \left(\frac{\ell(\gamma)}{4}\right)^2$

$\Longrightarrow$ the result. ∎

Some open questions:

- Generalize this to closed surfaces? Probably not...

- How does $\mathcal{B}(S)$ compare with $\operatorname{diam}(S)$?

Thanks to organizers + Scott for this great conference!