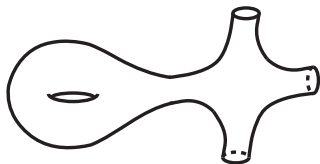


# Relative Hyperbolicity: Lost and Found

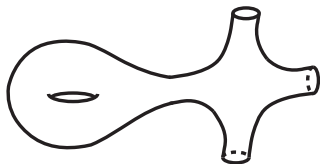
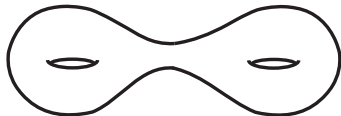
Jason Behrstock

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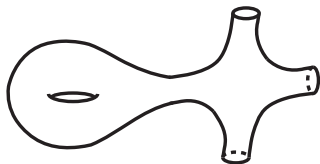
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We will only consider the cases where  $3g + p - 3 > 1$ .

# Hyperbolicity

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## Theorem (Masur–Minsky)

*$MCG(S)$  is weakly relatively hyperbolic with respect to a finite collection of multicurve stabilizers (and their cosets).*

## “Definition”

Fix a metric space  $(X, \text{dist})$ . Define the *asymptotic cone* to be:

$$\text{Cone}(X, \text{dist}) = \lim_{n \rightarrow \infty} (X, \frac{1}{n} \text{dist})$$

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### Examples:

- ▶  $\text{Cone}(\mathbb{Z}^m, \text{Manhattan}) = (\mathbb{R}^m, \ell^1)$
- ▶ The asymptotic cone of a  $\delta$ -hyperbolic space is an  $\mathbb{R}$ -tree.

## Definition

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## Remark

*There is a unique finest such structure on any given space.*

## Theorem (Druţu–Sapir)

*$G$  is relatively hyperbolic with respect to a collection of subsets  $\mathcal{H}$  if and only if every asymptotic cone of  $G$  is tree-graded with respect to subsets of the form  $\text{Cone}(H)$ , where  $H \in \mathcal{H}$ .*

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## Theorem (Behrstock)

*$\text{Cone}(\mathcal{MCG}(S))$  contains cut-points.*

Goal: Understand the pieces in the tree-graded structure on  $\text{Cone}(\mathcal{MCG})$ .

## Theorem (Behrstock–Druţu–Mosher)

*There is no collection of subsets,  $\mathcal{H}$ , with respect to which  $\text{Cone}(\mathcal{MCG}(S))$  is tree-graded with respect to  $\text{Cone}(\mathcal{H})$ .*

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## Remark

*Hence,  $\mathcal{MCG}(S)$  is not relatively hyperbolic as a group or even as a metric space. That it is not relatively hyperbolic with respect to a collection of subgroups  $\mathcal{H}$  was proven independently by Anderson–Aramayona–Shackleton.*

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Behrstock–Druţu–Mosher)

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1. If  $3g + p - 3 < 3$ , then  $\mathcal{T}$  is  $\delta$ -hyperbolic.
2. If  $3g + p - 3 = 3$ , then  $\mathcal{T}$  is relatively hyperbolic.
3.  $3g + p - 3 > 3$ , then there is no collection of subsets,  $\mathcal{H}$ , with respect to which  $\text{Cone}(\mathcal{T})$  is tree-graded with respect to  $\text{Cone}(\mathcal{H})$ .

## Description of pieces

### Theorem (Behrstock–Kleiner–Minsky–Mosher)

Suppose that  $\xi(S) \geq 2$ . For any pair of points  $\mu, \nu \in \text{Cone}(\mathcal{MCG}(S))$ , the following are equivalent:

1. No point of  $\text{Cone}(\mathcal{MCG}(S))$  separates  $\mu$  from  $\nu$ .
2. In any neighborhoods of  $\mu, \nu$ , respectively, there exist points  $\mu', \nu'$  with representative sequences  $(\mu'_n), (\nu'_n)$  such that

$$\lim d_{\mathcal{C}(S)}(\mu'_n, \nu'_n) < \infty$$

## Curve Complex

The **curve complex of  $S$** ,  $\mathcal{C}(S)$ , is the simplicial complex with a **vertex** for each homotopy class of non-trivial and non-peripheral simple closed curve on  $S$ . And an **edge** for each pair of vertices corresponding to curves which can be realized disjointly.

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