Calculus II – Solutions to Sample Problems

Math 221

Spring 2010

1 Areas of Surfaces of Revolution Section 6.4

In problems 1 – 8 of this section you don't have to evaluate the integral by hand (most of them are not so easy). If you have a calculator that has numerical integration then use it to get the answer. The TI 83 for example has numerical integration. It is under $\boxed{\text{Math}}$. For example, to do $\int_0^3 x^2\,dx$ using TI - 83 do: $\boxed{\text{Math}}$ $\boxed{fnInt(x^2, x, 0, 3)}$. Observe that the arguments for fnInt are: The function to be integrated, the variable of integration, the lower limit , and the upper limit of integration.

If your calculator does not do definite integrals check if the integral can be computed using the table of integration given at the end of the book. If you don't find the integral there then just leave the integral indicated.

To compute the numerical value of Problem 7 my advice is that you compute the integral $\int_0^y \tan t\,dt$ explicitly. If you don't do that you probably would need a more powerful calculator or a computer program like Mathematica or Maple. For now it is ok to just leave the integral indicated.

The following problem is similar to Problem 7.

Problem 8. In this problem the graph of the function $y = \int_1^x \sqrt{t^2 - 1}\,dt$ is revolved about the x-axis and we have to compute the surface area for x between 1 and $\sqrt{5}$. Notice the the function y is given as an integral so to compute its derivative we will use the Fundamental Theorem of Calculus:

$$y' = \sqrt{x^2-1}.$$

We will need $\sqrt{1+y'^2}$ so let us compute it right now:

$$1+y'^2 = 1+(x^2-1) = x^2$$

Ok, so

$$\sqrt{1+y'^2} = \sqrt{x^2} = x$$

because $x \geq 0$.

The surface area is

$$S = 2\pi \int_1^{\sqrt{5}} y\sqrt{1+(y')^2}\,dx = 2\pi \int_1^{\sqrt{5}} x \int_1^x \sqrt{t^2-1}\,dt\,dx \qquad (1)$$

At this point it would be ok to leave the integral indicated. But if we want to complete the exercise then we look at the table T-2. Formula 37 that says

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C$$

Therefore,

$$\int_1^x \sqrt{t^2-1}\,dx = \frac{t}{2}\sqrt{t^2-1} - \frac{1}{2}\ln|t+\sqrt{t^2-1}|\Big|_1^x = \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln|x+\sqrt{x^2-1}|.$$

Substituting this into Equation 1 we obtain

$$S = 2\pi \int_1^{\sqrt{5}} x\left[\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln|x+\sqrt{x^2-1}|\right]\,dx.$$

Using a TI 84 I got 8.5469 as approximate answer. Please let me know if there is a mistake.

Problem 10. This problem refers to a cone that is generated by a line segment that rotates about the y-axis. The line segment is given by $y = x/2$ from $x = 0$ to $x = 4$. Observe that because the rotation is about the y-axis, we will have an integral in the y direction. The y values will vary from $y = 0$ (that corresponds to $x = 0$) to $y = 2$ (that corresponds to $x = 4$). The radius of rotation for every value of y is given by $x = 2y$. The formula for the surface area to be used here is

$$S = 2\pi \int_0^2 x \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy,$$

where $x = 2y$. So we get

$$S = 2\pi \int_0^2 2y \sqrt{(2)^2 + 1}\, dy = 2\pi\sqrt{5} \cdot y^2 \Big|_0^2 = 2\pi\sqrt{5} \cdot 4 = 8\pi\sqrt{5}.$$

As explained in class, the surface area of a right circular cone is given by $\pi R l$, where R is the radius of the base and l is the slant height (the distance from the vertex to any point on the border of the base). For the cone of this problem R is the x value when $y = 2$, that is 4, and l is computed by Pythagorean theorem as the hypothenuse of a right triangle with legs 2 and 4. Therefore, $l = \sqrt{20} = 2\sqrt{5}$. It follows that the surface area of the cone is $\pi \cdot 4 \cdot 2\sqrt{5} = 8\pi\sqrt{5}$ which is the same answer we obtained by integration.

We have finished the problem.

Problem 14 Here the radius of rotation for each value of x is $y = \sqrt{x}$. The formula for the surface area to be used here is

$$S = 2\pi \int_{3/4}^{15/4} y \sqrt{\left(1 + \frac{dy}{dx}\right)^2}\, dx, \tag{2}$$

where $y = \sqrt{x}$. At this point we all should now cold that $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$ because we have computed it several times. Therefore,

$$\sqrt{1 + y'^2} = \sqrt{1 + \frac{1}{4x}} = \sqrt{\frac{4x + 1}{4x}} = \frac{1}{2}\sqrt{\frac{1 + 4x}{x}}.$$

Substituting into Equation 2 we obtain

$$S = 2\pi \int_{3/4}^{15/4} \sqrt{x} \cdot \frac{1}{2}\sqrt{\frac{1 + 4x}{x}}\, dx = \pi \int_{3/4}^{15/4} \sqrt{1 + 4x}\, dx.$$

For the last integral we use the substitution $u = 1+4x$; $du = 4\,dx$; $u(3/4) = 4$; $u(15/4) = 16$. Then

$$S = \pi \int_4^{16} \sqrt{u}\,\frac{du}{4} = \frac{\pi}{4}\,\frac{u^{3/2}}{3/2}\Big|_4^{16} = \frac{\pi}{6}\,[64-8] = \frac{56\pi}{6} = \frac{28\pi}{3}.$$

We have finished the problem.

Problem 18 In this problem we will experience a situation when the function is negative and therefore the radius of rotation is not the function given but its opposite value. You may not notice this at first and you start doing computations as usual. At the end you will get a negative value for the surface area. This of course does not make sense and is a sign that something went wrong. It is convenient to graph the function first to see if some adjustments need to be made to the regular formulas.

I will solve the problem mechanically (making a mistake) and at the end I will fix the answer.

The formula for the surface area to be used here is

$$S = 2\pi \int_1^3 x \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\; dy, \tag{3}$$

where $x = (1/3)y^{3/2} - y^{1/2}$. (Here is the mistake because the radius of rotation in this case is not x which is negative, but $-x$)

$$\frac{dx}{dy} = (1/2)y^{1/2} - (1/2)y^{-1/2} = \frac{1}{2}\left[y^{1/2} - y^{-1/2}\right]$$

$$\left(\frac{dx}{dy}\right)^2 + 1 = \frac{1}{4}\left[y - 2 + y^{-1} + 4\right] = \frac{1}{4}\left[y + 2 + y^{-1}\right] = \frac{1}{4}\left[y^{1/2} + y^{-1/2}\right]^2.$$

Hence,

$$\sqrt{\left(\frac{dx}{dy}\right)^2 + 1} = \frac{1}{2}\left[y^{1/2} + y^{-1/2}\right].$$

Substituting into Equation 3 we obtain,

$$S = 2\pi \int_1^3 \left((1/3)y^{3/2} - y^{1/2}\right)\cdot\frac{1}{2}\left(y^{1/2} + y^{-1/2}\right)\,dy = \pi \int_1^3 \left[(1/3)y^2 + (1/3)y - y - 1\right] dy$$

$$= \pi \int_1^3 \left[(1/3)y^2 - (2/3)y - 1\right] dy = \pi \left[(1/9)y^3 - (1/3)y^2 - y\right]_1^3 = [-3 - \frac{-11}{9}]\pi = -\frac{38}{9}\pi.$$

BUT WAIT! The surface area cannot be negative. Therefore a mistake has been made. Fortunately in this case the mistake is just a sign due to the fact that the given function is negative. We just need to change the sign. The correct answer to the problem is then $\frac{38\,\pi}{9}$.

We have finished the problem.

2 Moments and Centers of Mass Section 6.7

Problem 8. When the density is constant the center of mass does not depend on the density (and it is called the centroid). Therefore we can assume that the density is 1. If we draw the region we see that it is symmetric with respect to the y-axis (indeed, the function $\sec^2 x$ is even so its graph is symmetric with respect to the y-axis, and the interval $-\pi/4 \leq x \leq \pi/4$ is also symmetric with respect to the y-axis). Therefore, the centroid lies on the $y-$ axis. This means that $\overline{x} = 0$ and we need to compute just $\overline{y}$.

$$\overline{y} = \frac{M_x}{M} = \frac{\int_{-\pi/4}^{\pi/4} \tilde{y}\, dm}{\int_{-\pi/4}^{\pi/4} dm}$$

In this problem $\tilde{y} = \frac{\sec^2 x}{2}$ and $dm = dA = \sec^2 x\, dx$ (we are assuming $\delta = 1$).

Let us compute the parts. The mass is easier to compute:

$$M = \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \tan x \Big|_{-\pi/4}^{\pi/4} = 1 - (-1) = 2.$$

For the second one the knowledge of the trigonometric identity $\sec^2 x = \tan^2 x + 1$ and some experience with integration will help:

$$M_x = \int_{-\pi/4}^{\pi/4} \frac{\sec^2 x}{2} \cdot \sec^2 x\, dx = \frac{1}{2} \int_{-\pi/4}^{\pi/4} (\tan^2 x + 1) \cdot \sec^2 x\, dx.$$

We apply the substitution $u = \tan x$; $du = \sec^2 x$; $u(-\pi/4) = -1$; $u(\pi/4) = 1$ to obtain

$$M_x = \int_{-1}^{1} (u^2 + 1)\, du = 2 \int_{0}^{1} (u^2 + 1)\, du = (u^3/3 + u)\Big|_0^1 = 1/3 + 1 = 4/3.$$

So $\overline{y} = \frac{4/3}{2} = 2/3$.

The answer is: The center of mass (centroid) is the point $(0, 2/3)$.

Problem 12. First we graph the region. We have to find the points of intersection of the two parabolas. So we solve the equation $2x^2 - 4x = 2x - x^2$. This is equivalent to $3x^2 - 6x = 0$ or $x(x-2) = 0$. The solutions are $x = 0$, and $x = 2$ (the correspondent y values are both zero).

Observe that the axis of symmetry for each of the two parabolas is the line $x = 1$ (use the formula $x = -b/(2a)$ from algebra). So the region is symmetric with respect to the line $x = 1$. This implies that $\overline{x} = 1$ and we need to compute just $\overline{y}$.

Recall that $dm = dA$ because we assume $\delta = 1$. In this case $dA = (\text{top} - \text{bottom})\, dx = [(2x - x^2) - (2x^2 - 4x)]\ dx = (-3x^2 + 6x)\, dx$.

$$\tilde{y} = \frac{(2x^2 - 4x) + (2x - x^2)}{2} = \frac{x^2}{2} - x.$$

$$M = \int_0^2 (-3x^2 + 6x)\, dx = (-x^3 + 3x^2)\Big|_0^2 = -8 + 12 = 4.$$

$$M_x = \int_0^2 \tilde{y}\, dm = \int_0^2 \left(\frac{x^2}{2} - x\right) (-3x^2 + 6x)\, dx = \frac{1}{2} \int_0^2 \left(x^2 - 2x\right) (-3x^2 + 6x)\, dx$$

$$= \frac{1}{2} \int_0^2 -3x^4 + 12x^3 - 12x^2\, dx = \frac{1}{2} \left[-(3/5)x^5 + 3x^4 - 4x^3\right]_0^2 = \frac{1}{2}(-\frac{96}{5} + 48 - 32) = -1.6$$

$$\overline{y} = \frac{M_x}{M} = \frac{-1.6}{4} = -0.4$$

The answer is: The centroid is the point $(1, -0.4)$.

Problem 18. In this problem the density δ is variable but depends only on one variable. The students who are struggling with the course may skip the problems with variable density and concentrate in the problems that ask for the center of mass when the density is constant. If the density depends on two variables then the computation of the center of mass requires double integrals that are studied in Calculus III.

(a)

$$V = \pi \int_1^4 \left(\frac{2}{x}\right)^2 dx = 4\pi \left(\frac{-1}{x}\right)\Big|_1^4 = -4\pi(\frac{1}{4} - 1) = 3\pi$$

(b) Because the density is variable we cannot assume that it is equal to 1. We compute the parts that we need to compute the center of mass of the thin plate:

$$dm = \delta\, dA = \sqrt{x}\,\frac{2}{x}\,dx = \frac{2}{\sqrt{x}}\,dx$$

$$\tilde{y} = \frac{1}{x}, \qquad \tilde{x} = x$$

$$M = \int_1^4 dm = \int_1^4 \frac{2}{\sqrt{x}}\,dx = 4\sqrt{x}\Big|_1^4 = 4(2-1) = 4.$$

$$\begin{aligned} M_x = \int_1^4 \tilde{y}\,dm &= \int_1^4 \frac{1}{x}\,\frac{2}{\sqrt{x}}\,dx = \int_1^4 \frac{2}{x^{3/2}}\,dx = 2\int_1^4 x^{-3/2}\,dx \\ &= -4x^{-1/2}\Big|_1^4 = -4(1/2-1) = 2. \end{aligned}$$

$$M_y = \int_1^4 \tilde{x}\,dm = \int_1^4 x\frac{2}{\sqrt{x}}\,dx = 2\int_1^4 x^{1/2}\,dx = 2\cdot\frac{2}{3}\cdot x^{3/2}\Big|_1^4 = \frac{4}{3}(8-1) = 28/3.$$

$$\overline{x} = \frac{M_y}{M} = \frac{28/3}{4} = \frac{7}{3}, \qquad \overline{y} = \frac{M_x}{M} = \frac{2}{4} = \frac{1}{2}.$$

The answer is: The center of mass of the plate is the point $(\frac{1}{2}, \frac{7}{3})$.

Problem 24. This problem is based on Problem 21. Please do Problem 21 first. Then the answer to Problem 24 is immediate:

$$\overline{x} = \frac{a}{3}, \quad \overline{y} = \frac{a}{3}.$$

3 Integration by Parts
Section 7.1

Problem 6. To compute $I = \int_1^e x^3 \ln x \, dx$ by parts we have to decide which part is going to be u and which dv. Since the integral of $\ln x$ is something more complicated than $\ln x$ the choice must be:

$$\begin{aligned} u &= \ln x & \qquad dv &= x^3 \, dx \\ du &= \tfrac{1}{x} \, dx & \qquad v &= \frac{x^4}{4} \end{aligned}$$

We get

$$I = \ln x \cdot \frac{x^4}{4} \Big|_1^e - \int_1^e \frac{x^4}{4} \frac{1}{x} \, dx = \frac{e^4}{4} - \int_1^e \frac{x^3}{4} \, dx = \frac{e^4}{4} - \frac{x^4}{16} \Big|_1^e = \frac{e^4}{4} - \left(\frac{e^4}{16} - \frac{1}{16} \right) = \frac{3e^4 - 1}{16}.$$

Problem 10. To compute $I = \int 4x \sec^2 2x \, dx = 4 \int x \sec^2 2x \, dx$ we can use

$$\begin{aligned} u &= x & \qquad dv &= \sec^2 2x \, dx \\ du &= dx & \qquad v &= \frac{\tan 2x}{2} \end{aligned}$$

So

$$I = 4 \left[x \frac{\tan 2x}{2} - \int \frac{\tan 2x}{2} \, dx \right]$$

At this point of the course you should have computed the integral of $\tan x$ more than once and you should know why $\int \tan u \, du = -\ln |\cos u| + C$. From here with the help of a simple substitution you should be able to figure out that

$$\int \tan(2x) \, dx = -\frac{1}{2} \ln |\cos 2x| + C.$$

Therefore,

$$I = 4 \left[x \frac{\tan 2x}{2} - \frac{1}{2} \int \tan(2x) \, dx \right] = 4 \left[x \frac{\tan 2x}{2} + \frac{1}{4} \ln |\cos 2x| \right] + C$$

that simplifies to

$$I = 2x \tan(2x) + \ln|\cos(2x)| + C.$$

Problem 14. To compute $I = \displaystyle\int (r^2 + r + 1)e^r \, dr$ we will need to integrate by parts a few times. The natural choice for u is the polynomial $r^2 + r + 1$.

$$\begin{aligned} u &= r^2 + r + 1 & dv &= e^r \, dr \\ du &= (2r + 1) \, dr & v &= e^r \end{aligned}$$

$$I = (r^2 + r + 1)e^r - \int (2r + 1)e^r \, dr = (r^2 + r + 1)e^r - J, \tag{4}$$

where $J = \int (2r + 1)e^r \, dr$.

To compute J we use

$$\begin{aligned} u &= 2r + 1 & dv &= e^r \, dr \\ du &= 2 \, dr & v &= e^r \end{aligned}$$

So

$$J = (2r + 1)e^r - \int 2e^r \, dr = (2r + 1)e^r - 2e^r + C = e^r(2r - 1) + C. \tag{5}$$

Putting (4) and (5) together we get

$$I = (r^2 + r + 1)e^r - J = (r^2 + r + 1)e^r - e^r(2r - 1) + C = e^r \left[r^2 - r + 2\right] + C.$$

Problem 26. As you probably noticed your success in integration depends heavily on your algebra skills. In this problem you are asked to do a substitution before you do integration by parts in order to compute $I = \int_0^1 \ln(x + x^2) \, dx$. However, no substitution seems to improve the looks of the integral the way it is now. Here is where we can use our algebra and precalculus knowledge. We can rewrite

$$\ln(x + x^2) = \ln(x(x + 1) = \ln x + \ln(x + 1).$$

The integral of $\ln x$ was done in class and is also Example 2 in the book.

$$\int \ln x \, dx = x \, \ln x - x + C$$

For the integral of $\ln(x+1)$ we can do a simple substitution $u = x+1;\ du = dx$.

$$\int \ln(x+1)\, dx = \int \ln(u)\, du = u \, \ln u - u + C = (x+1) \, \ln(x+1) - (x+1) + C.$$

So

$$I = x \, \ln x - x + (x+1) \, \ln(x+1) - (x+1) + C = x \, \ln x + (x+1) \, \ln(x+1) - 2x - 1 + C.$$

There are many ways in which the answer can be rewritten. For example $-1+C$ is basically the same as C because C is an arbitrary constant. Also

$$x \ln x + x \ln(x+1) = x \ln(x(x+1)) = x \ln(x^2+x).$$

So the answer can be written as

$$I = x \ln(x^2+x) + \ln(x+1) + C.$$

There is actually another way of writing the answer so that it uses only one logarithm:

$$I = \ln\big((x^2+x)^x (x+1)\big) + C.$$

Problem 40. To integrate $I_n = \int x^n \sin x \, dx$ by parts we use

$$\begin{aligned} u &= x^n & dv &= \sin x \, dx \\ du &= nx^{n-1}\, dx & v &= -\cos x \end{aligned}$$

and get

$$I = -x^n \cos x - \int (-\cos x) n x^{n-1}\, dx = -x^n \cos x + n \int (\cos x) x^{n-1}\, dx = -x^n \cos x + n I_{n-1} dx.$$

4 Trigonometric Integrals
Section 7.2

Problem 6. Because the exponent of the cosine is odd the substitution $u = \sin t$; $du = \cos t\, dt$; $u(0) = 0$; $u(\frac{\pi}{2}) = 1$ must work. From $\cos^7 t$ one cosine is used for du and the six remaining can be expressed in terms of sine as follows:

$$\cos^6 t = (\cos^2 t)^3 = (1 - \sin^2 t)^3.$$

We will also need the formula to expand the cube of a binomial: $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$. So we have

$$\int_0^{\pi/2} 7\cos^7 t\, dt = 7\int_0^1 \left(1-u^2\right)^3 du = 7\int_0^1 \left(1 - 3u^2 + 3u^4 - u^6\right) du$$

$$= 7\left[u - u^3 + \frac{3}{5}u^5 - \frac{u^7}{7}\right]_0^1 = 7\left(1 - 1 + \frac{3}{5} - \frac{1}{7}\right) = \frac{21}{5} - 1 = \frac{16}{5}.$$

Problem 12. Now the exponent of the sine is the one that is odd.

Let $u = \cos 2x$; $du = -2\sin 2x\, dx$; $u(0) = 1$; $u(\pi) = 1$. Observe that we have a very special circumstance here. Namely, the two limits of integration after the substitution are the same! ($u(0) = 1$; $u(\pi) = 1$). Therefore the answer to the problem is 0 and in principle we don't have to perform any computations. However, for the sake of practice let us compute the indefinite integral.

$$\int \sin 2x \, \cos^2 2x\, dx = -\frac{1}{2}\int \cos^2 2x\,(-2\sin 2x)\, dx = -\frac{1}{2}\int u^2\, du = -\frac{1}{2}\frac{u^3}{3} + C$$

$$= -\frac{1}{2}\frac{\cos^3 2x}{3} + C = -\frac{\cos^3 2x}{6} + C.$$

Observe that if we were to solve the original problem by evaluating the antiderivative $-\frac{\cos^3 2x}{6}$ from 0 to π we would obtain 0 because the antiderivative has the same values at 0 and at π.

Problem 18. Here we have to use a trigonometric identity to turn the radicand (the expression inside the radical) into a perfect square so that the square root gets eliminated. The appropriate identity is

$$1 - \cos^2 2\theta = \sin^2 2\theta.$$

When we eliminate the square root we have to be very careful with the signs. It is not true that $\sqrt{x^2} = x$ in general. If x is negative then $\sqrt{x^2}$ is actually $-x$. In general, $\sqrt{x^2} = |x|$.

In our problem we need to simplify $\sqrt{\sin^2\theta}$, where θ varies from 0 to π. In this interval the $\sin\theta$ is nonnegative. Therefore, $\sqrt{\sin^2\theta} = \sin\theta$.

$$\int_0^\pi \sqrt{1-\cos^2\theta}\,d\theta = \int_0^\pi \sqrt{\sin^2\theta}\,d\theta = \int_0^\pi \sin\theta\,d\theta = -\cos\theta\,\Big|_0^\pi = -(-1-1) = 2.$$

Problem 28. For this problem we need to keep in mind the relationships between $\csc x$ and $\cot x$ in terms of derivatives and identities. It would be useful to know that $\csc^2 x = 1 + \cot^2 x$ and that $(\cot x)' = -\csc^2 x$. In our integral we have the fourth power of $\csc x$ (for the angle $x = \frac{\theta}{2}$). The substitution that will work is

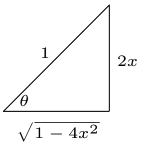
$$u = \cot\frac{\theta}{2};\ du = -\frac{1}{2}\csc^2\frac{\theta}{2}\,d\theta;\ u(\pi/2) = 1;\ u(\pi) = 0.$$

$$\int_{\pi/2}^{\pi} 3\csc^4\frac{\theta}{2}\,d\theta = 3\int_{\pi/2}^{\pi} \csc^4\frac{\theta}{2}\,d\theta = 3\int_{\pi/2}^{\pi} \csc^2\frac{\theta}{2}\csc^2\frac{\theta}{2}\,d\theta$$

$$= 3\cdot(-2)\int_{\pi/2}^{\pi}\left(1+\cot^2\frac{\theta}{2}\right)\left(-\frac{1}{2}\csc^2\frac{\theta}{2}\right)d\theta = -6\int_1^0 (1+u^2)\,du = -6\left[u+\frac{u^3}{3}\right]_1^0 = 8.$$

5 Trigonometric Substitutions
Section 7.3

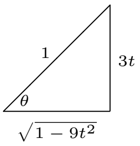
Problem 6. The key expression in the integral is $\sqrt{1-4x^2}$. To find the appropriate substitution we produce a right triangle with hypotenuse 1, and legs $2x$ and $\sqrt{1-4x^2}$.



$$x = \tfrac{1}{2}\sin\theta; \quad dx = \tfrac{1}{2}\cos\theta\, d\theta$$
$$\sqrt{1-4x^2} = \cos\theta$$
$$\theta = \sin^{-1}(2x); \qquad \theta(0) = 0; \quad \theta(\tfrac{1}{2\sqrt{2}}) = \tfrac{\pi}{4}$$

$$\int_0^{\frac{1}{2\sqrt{2}}} \frac{2\, dx}{\sqrt{1-4x^2}} = \int_0^{\frac{\pi}{4}} \frac{\cos\theta\, d\theta}{\cos\theta} = \int_0^{\frac{\pi}{4}} d\theta = \frac{\pi}{4}.$$

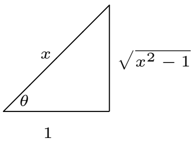
Problem 8. For this problem the hypotenuse is 1 and the legs $3t$ and $\sqrt{1-9t^2}$ respectively.



$$t = \tfrac{1}{3}\sin\theta; \quad dt = \tfrac{1}{3}\cos\theta\, d\theta$$
$$\sqrt{1-9t^2} = \cos\theta$$
$$\theta = \sin^{-1}(3t)$$

$$\int \sqrt{1-9t^2}\, dt = \frac{1}{3}\int \cos^2\theta\, d\theta = \frac{1}{3}\int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{3}\cdot\frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$
$$= \frac{1}{6}\left[\theta + \sin\theta\cos\theta\right] + C = \frac{1}{6}\left[\sin^{-1}(3t) + 3t\sqrt{1-9t^2}\right] + C.$$

Problem 14. Now the hypotenuse is x. We know that because the minuend in the radicand is x^2.



$$x = \sec\theta; \quad dx = \sec\theta\tan\theta\, d\theta$$
$$\sqrt{x^2-1} = \tan\theta$$
$$\theta = \sec^{-1}(x)$$

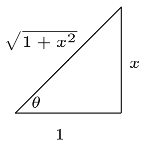
$$\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = 2\int \frac{\sec\theta\tan\theta\,d\theta}{\sec^3\theta\tan\theta} = 2\int \cos^2\theta\,d\theta = \theta + \frac{\sin 2\theta}{2} + C$$

$$= \theta + \sin\theta\cos\theta + C = \sec^{-1}x + \frac{\sqrt{x^2-1}}{x}\cdot\frac{1}{x} + C = \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C.$$

Problem 30. Let $x = e^t$; $dx = e^t\,dt$; $x(\ln(\frac{3}{4})) = \frac{3}{4}$; $x(\ln(\frac{4}{3})) = \frac{4}{3}$. Then

$$I = \int_{\ln\frac{3}{4}}^{\ln\frac{4}{3}} \frac{e^t\,dt}{(1+e^{2t})^{\frac{3}{2}}} = \int_{\frac{3}{4}}^{\frac{4}{3}} \frac{dx}{(1+x^2)^{\frac{3}{2}}}.$$

Now we apply a trigonometric substitution



$x = \tan\theta$; $\quad dx = \sec^2\theta\,d\theta$

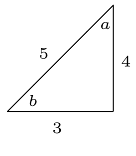
$\sqrt{1+x^2} = \sec\theta$

$\theta = \tan^{-1}(x)$

For the moment let us say that $\tan^{-1}(\frac{3}{4}) = a$ and that $\tan^{-1}(\frac{4}{3}) = b$. Then

$$I = \int_a^b \frac{\sec^2\theta\,d\theta}{sec^3\theta} = \int_a^b \frac{d\theta}{sec\theta} = \int_a^b \cos\theta\,d\theta = \sin\theta\Big|_a^b = \sin b - \sin a.$$

To simplify the last expression it is convenient to use a right triangle (the $3, 4, 5$ right triangle)



We see that $b = \tan^{-1}\frac{4}{3}$; $\quad \sin b = \frac{4}{5}$

Likewise, $a = \tan^{-1}\frac{3}{4}$; $\quad \sin a = \frac{3}{5}$

Therefore, $I = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$.

6 Partial Fractions
Section 7.4

Problem 2. We factor the denominator as

$$x^2 - 3x + 2 = (x-2)(x-1).$$

Then according to the general recipe

$$\frac{5x-7}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{A(x-1)+B(x-2)}{(x-2)(x-1)}.$$

Equating the numerators we get

$$5x - 7 = A(x-1) + B(x-2). \tag{6}$$

To determine the (undetermined) coefficients A, B we plug into Equation 6 nice values of x, namely $x = 1$ and $x = 2$.

With $x = 2$ we obtain: $3 = A$; With $x = 1$ we get: $-2 = -B$ so $B = 2$.

The answer is:

$$\frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}.$$

Problem 12. Factoring the denominator and following the general recipe we have:

$$I = \int \frac{2x+1}{x^2-7x+12}\,dx = \int \frac{A}{x-6}\,dx + \frac{B}{x-1}\,dx = A\ln|x-6| + B\ln|x-1| + K,$$

where the coefficients A and B are going to be found next, and K is the constant of integration.

$$\frac{2x+1}{x^2-7x+12} = \frac{A}{x-6} + \frac{B}{x-1} = \frac{A(x-1)+B(x-6)}{(x-1)(x-6)}$$

so

$$2x + 1 = A(x-1) + B(x-6).$$

Setting $x = 1$ we get $3 = -5B$, and setting $x = 6$ we get $13 = 5A$. Therefore, $A = \frac{13}{5}$ and $B = -\frac{3}{5}$.

The answer is

$$I = \frac{13}{5}\ln|x-6| - \frac{3}{5}\ln|x-1| + K,$$

where K is the constant of integration.

Problem 20. The denominator can be factored further.

$$(x-1)(x^2+2x+1) = (x-1)(x+1)^2.$$

Observe that $(x+1)^2$ should be seen as the linear factor $(x+1)$ with multiplicity 2 and therefore it gives rise to two partial (elementary) fractions.

$$\frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2} = \frac{A(x+1)^2+B(x-1)(x+1)+C(x-1)}{(x-1)(x+1)^2}$$

where A, B, C can be found from the identity

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1). \tag{7}$$

Setting $x=-1$ we get $1=-2C$ so $C=-\frac{1}{2}$. Setting $x=1$ we get $1=4A$ so $A=\frac{1}{4}$.

To find B we compare the coefficients of x^2. The right hand side expanded contains Ax^2 and Bx^2 as the only terms with x^2 in them. Therefore, $1 = A+B$ from where we get $B = 1-A = 1-\frac{1}{4} = \frac{3}{4}$.

$$\int \frac{x^2}{(x-1)(x^2+2x+1)}\,dx = \int \frac{A}{x-1}\,dx + \int \frac{B}{x+1}\,dx + \int \frac{C}{(x+1)^2}\,dx$$

$$= A\ln|x-1| + B\ln|x+1| - \frac{C}{x+1} + K,$$

where A, B, C are the constants found above and K is the constant of integration.

Problem 36. To compute

$$I = \int \frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt$$

we first do the substitution $u=e^t$; $du = e^t\,dt$ Observe that the numerators of the integrand factors as $e^t(e^{3t}+2e^t-1)$. Hence,

$$I = \int \frac{u^3+2u-1}{u^2+1}\,du.$$

The integrand is not a proper fraction. We do a long division to obtain

$$\frac{u^3+2u-1}{u^2+1} = u + \frac{u-1}{u^2+1}$$

(actually if you think cleverly you don't really need to do the long division. Just observe that if you multiply the denominator u^2+1 by u you get something close to the numerator, $u-1$ being the remainder.)

Observe now that $\frac{u-1}{u^2+1}$ is a partial fraction itself (linear over quadratic not factorable). Let us compute its integral first.

$$\int \frac{u-1}{u^2+1}\,du = \int \frac{u}{u^2+1}\,du - \int \frac{1}{u^2+1}\,du = \frac{1}{2}\ln(u^2+1) - \tan^{-1}(u) + C.$$

Putting things together,

$$\begin{aligned} I = \int (u + \frac{u-1}{u^2+1})\,du &= \frac{u^2}{2} + \frac{1}{2}\ln(u^2+1) - \tan^{-1}(u) + C \\ &= \frac{e^{2t}}{2} + \frac{1}{2}\ln(e^{2t}+1) - \tan^{-1}(e^t) + C. \end{aligned}$$