Real Analysis II

Closed Subsets of Complete Metric Spaces are Complete

- (X, ρ) – Complete Metric Space.

- (X, ρ) – Complete Metric Space.
- X is called complete if every Cauchy sequence in X converges to an element $x \in X$.

- (X, ρ) – Complete Metric Space.
- X is called complete if every Cauchy sequence in X converges to an element $x \in X$.
- (Y, ρ) metric subspace of (X, ρ).

- (X, ρ) – Complete Metric Space.
- X is called complete if every Cauchy sequence in X converges to an element $x \in X$.
- (Y, ρ) metric subspace of (X, ρ).
- Y is complete if and only if Y is closed in X.

- (X, ρ) – Complete Metric Space.
- X is called complete if every Cauchy sequence in X converges to an element $x \in X$.
- (Y, ρ) metric subspace of (X, ρ).
- Y is complete if and only if Y is closed in X.
- If Y is complete and x is a point of closure for Y then there exists a sequence $\{y_n\}$ in Y such that $y_n \to x$ in X. But then $\{y_n\}$ is Cauchy in Y and therefore it must converge to an element of Y. It follows that x is an element of Y. Therefore, Y is closed.

- (X, ρ) – Complete Metric Space.
- X is called complete if every Cauchy sequence in X converges to an element $x \in X$.
- (Y, ρ) metric subspace of (X, ρ).
- Y is complete if and only if Y is closed in X.
- If Y is complete and x is a point of closure for Y then there exists a sequence $\{y_n\}$ in Y such that $y_n \to x$ in X. But then $\{y_n\}$ is Cauchy in Y and therefore it must converge to an element of Y. It follows that x is an element of Y. Therefore, Y is closed.
- Let Y be closed in the complete metric space X. Let $\{y_n\}$ be a Cauchy sequence in Y. Since $\{y_n\}$ is Cauchy in X also, then it converges to an element $x \in X$. But since Y is closed x is in Y. Therefore, Y is complete.

- (X, ρ) – Complete Metric Space.

- (X, ρ) – Complete Metric Space.
- $E \subseteq X$. The diameter of E is the number $\mathrm{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\}$.

- (X, ρ) – Complete Metric Space.
- $E \subseteq X$. The diameter of E is the number $\mathrm{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\}$.
- A sequence $\{E_n\}$ of sets is called *descending* if $E_{n+1} \subseteq E_n$ for all n.

- (X, ρ) – Complete Metric Space.
- $E \subseteq X$. The diameter of E is the number $\mathrm{diam}(E) = \sup\{\rho(x,y) \mid x, y \in E\}$.
- A sequence $\{E_n\}$ of sets is called *descending* if $E_{n+1} \subseteq E_n$ for all n.

- Theorem
  (Cantor's Intersection Theorem) Let $\{E_n\}$ be a descending sequence of nonempty closed subsets of the complete metric space X, such that $\lim_{n\to\infty} \mathrm{diam}(E_n) = 0$. Then there exists a point $x \in X$ such that $\cap_{n=1}^{\infty} E_n = \{x\}$.

- (X, ρ) – Complete Metric Space.
- $E \subseteq X$. The diameter of E is the number $\mathrm{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\}$.
- A sequence $\{E_n\}$ of sets is called *descending* if $E_{n+1} \subseteq E_n$ for all n.

- Theorem
  (Cantor's Intersection Theorem) Let $\{E_n\}$ be a descending sequence of nonempty closed subsets of the complete metric space X, such that $\lim_{n\to\infty} \mathrm{diam}(E_n) = 0$. Then there exists a point $x \in X$ such that $\cap_{n=1}^{\infty} E_n = \{x\}$.
  - Choose an element x_k from each E_k. For every n the sequence $\{x_k\}_{k\geq n}^{\infty}$ is Cauchy sequence in the closed set E_n. So $\{x_k\}$ converges to an element $x \in E_n$. x is the only element contained in all the E_n.

- Let $a_n = 0.\underbrace{99...99}_{n} = 1 - 10^{-n}$, and $b_n = 1$. Then $\cap_{n=1}^{\infty}(a_n, b_n) = \emptyset$.

- Let $a_n = 0.\underbrace{99...99}_{n} = 1 - 10^{-n}$, and $b_n = 1$. Then $\bigcap_{n=1}^{\infty}(a_n, b_n) = \emptyset$.
- $\bigcap_{n=1}^{\infty}[n, \infty) = \emptyset$.

- (X, ρ) – metric space. K a subset of X.

- (X, ρ) – metric space. K a subset of X.
- $\{\mathcal{O}_i\}_{i \in I}$ a family of open subsets of X is called an *open covering of K* if $K \subseteq \cup_{i \in I} \mathcal{O}_i$.

- (X, ρ) – metric space. K a subset of X.
- $\{\mathcal{O}_i\}_{i\in I}$ a family of open subsets of X is called an *open covering of* K if $K \subseteq \cup_{i\in I}\mathcal{O}_i$.
- K is called *compact* if every open covering of K has a finite subcovering.

- (X, ρ) – metric space. K a subset of X.
- $\{\mathcal{O}_i\}_{i \in I}$ a family of open subsets of X is called an *open covering of* K if $K \subseteq \cup_{i \in I} \mathcal{O}_i$.
- K is called *compact* if every open covering of K has a finite subcovering.
- The closed intervals $[a, b]$ are compact subsets of $\mathbb{R}$.

- (X, ρ) – metric space. K a subset of X.
- $\{\mathcal{O}_i\}_{i \in I}$ a family of open subsets of X is called an *open covering of* K if $K \subseteq \cup_{i \in I} \mathcal{O}_i$.
- K is called *compact* if every open covering of K has a finite subcovering.
- The closed intervals $[a, b]$ are compact subsets of $\mathbb{R}$.
- The open intervals (a, b), with $a < b$ are not compact subsets of $\mathbb{R}$.

- (X, ρ) – metric space. K a subset of X.
- $\{\mathcal{O}_i\}_{i \in I}$ a family of open subsets of X is called an *open covering of* K if $K \subseteq \cup_{i \in I} \mathcal{O}_i$.
- K is called *compact* if every open covering of K has a finite subcovering.
- The closed intervals $[a, b]$ are compact subsets of $\mathbb{R}$.
- The open intervals (a, b), with $a < b$ are not compact subsets of $\mathbb{R}$.
- Closed and bounded subsets of $\mathbb{R}^n$ are compact subsets of $\mathbb{R}^n$.

- (X, ρ) – metric space. K a subset of X.
- $\{\mathcal{O}_i\}_{i \in I}$ a family of open subsets of X is called an *open covering of* K if $K \subseteq \cup_{i \in I} \mathcal{O}_i$.
- K is called *compact* if every open covering of K has a finite subcovering.
- The closed intervals $[a, b]$ are compact subsets of $\mathbb{R}$.
- The open intervals (a, b), with $a < b$ are not compact subsets of $\mathbb{R}$.
- Closed and bounded subsets of $\mathbb{R}^n$ are compact subsets of $\mathbb{R}^n$.
- The closed ball $B(0, 1) \subseteq L^p$ is not compact for any $p \in [1, \infty]$.

- Let $\mathcal{F}$ be a collection of subsets of a metric space X. We say that $\mathcal{F}$ has the *finite intersection property* if the intersection of any finite number of sets from $\mathcal{F}$ is nonempty.

- Let $\mathcal{F}$ be a collection of subsets of a metric space X. We say that $\mathcal{F}$ has the *finite intersection property* if the intersection of any finite number of sets from $\mathcal{F}$ is nonempty.
- $\mathcal{F}$ has the finite intersection property if and only if for any finite subcolection $\{F_1, \ldots, F_n\}$ of sets from $\mathcal{F}$, $\cup_{i=1}^{n} F_i' \neq X$.

- Let $\mathcal{F}$ be a collection of subsets of a metric space X. We say that $\mathcal{F}$ has the *finite intersection property* if the intersection of any finite number of sets from $\mathcal{F}$ is nonempty.
- $\mathcal{F}$ has the finite intersection property if and only if for any finite subcolection $\{F_1, \ldots, F_n\}$ of sets from $\mathcal{F}$, $\cup_{i=1}^{n} F_i' \neq X$.
- A metric space X is compact if and only if every collection $\mathcal{F}$ of closed subsets of X that has the finite intersection property has a nonempty intersection.

- Let $\mathcal{F}$ be a collection of subsets of a metric space X. We say that $\mathcal{F}$ has the *finite intersection property* if the intersection of any finite number of sets from $\mathcal{F}$ is nonempty.
- $\mathcal{F}$ has the finite intersection property if and only if for any finite subcolection $\{F_1, \ldots, F_n\}$ of sets from $\mathcal{F}$, $\cup_{i=1}^{n} F_i' \neq X$.
- A metric space X is compact if and only if every collection $\mathcal{F}$ of closed subsets of X that has the finite intersection property has a nonempty intersection.
- $(\Rightarrow)$ Let X be compact and $\mathcal{F}$ be a collection of closed subsets of $\mathcal{F}$ with the finite intersection property. Let $A = \cap\{F \,\Big|\, F \in \mathcal{F}\}$. $A' = \cup_{F\in\mathcal{F}} F'$. The F' form an open covering of A'. Since no finite subcollection $\{F'\}_{F\in\mathcal{F}}$ covers X, and X is compact, then $A' \neq X$. Therefore, $A \neq \emptyset$.

- ($\Leftarrow$) Let X be a metric space such that every collection of closed set that has the finite intersection property has a nonempty intersection. Let $\{\mathcal{O}_i\}_{i\in I}$ be any open covering of X. Assume that no finite subcollection of $\{\mathcal{O}_i\}_{i\in I}$ covers the whole space X. Then the family $\mathcal{F} = \{\mathcal{O}_i'\}_{i\in I}$ has the finite intersection property. Consequently, $\emptyset = (\cup_{i\in I}\mathcal{O}_i)' = \cap_{i\in I}\mathcal{O}_i' \neq \emptyset$. Contradiction.

- (X, ρ) – a metric space. K a subset of X.

- (X, ρ) – a metric space. K a subset of X.
- We say that K is totally bounded (subset of X) if for every $\varepsilon > 0$ there exists a finite set of balls of radius ε all centered in K such that K is contained in the union of those balls.

- (X, ρ) – a metric space. K a subset of X.
- We say that K is totally bounded (subset of X) if for every $\varepsilon > 0$ there exists a finite set of balls of radius ε all centered in K such that K is contained in the union of those balls.
- K is called bounded if there exists a ball $B(x, r)$ in X such that $K \subseteq B(x, r)$.

- (X, ρ) – a metric space. K a subset of X.
- We say that K is totally bounded (subset of X) if for every $\varepsilon > 0$ there exists a finite set of balls of radius ε all centered in K such that K is contained in the union of those balls.
- K is called bounded if there exists a ball $B(x, r)$ in X such that $K \subseteq B(x, r)$.
- Totally bounded implies bounded.

- (X, ρ) – a metric space. K a subset of X.
- We say that K is totally bounded (subset of X) if for every $\varepsilon > 0$ there exists a finite set of balls of radius ε all centered in K such that K is contained in the union of those balls.
- K is called bounded if there exists a ball $B(x, r)$ in X such that $K \subseteq B(x, r)$.
- Totally bounded implies bounded.
- The ball $B(0, 1)$ in L^2 is bounded but not totally bounded.

- (X, ρ) – a metric space.

- (X, ρ) – a metric space.
- X is said to have the Bolzano – Weierstrass Property if every infinite sequence has a cluster point.

- (X, ρ) – a metric space.
- X is said to have the Bolzano – Weierstrass Property if every infinite sequence has a cluster point.
- $x \in X$ is a cluster point for $\{x_n\}$ if every open neighborhood of x contains infinitely many terms of the sequence.

- (X, ρ) – a metric space.
- X is said to have the Bolzano – Weierstrass Property if every infinite sequence has a cluster point.
- $x \in X$ is a cluster point for $\{x_n\}$ if every open neighborhood of x contains infinitely many terms of the sequence.
- X is called sequentially compact if every sequence $\{x_n\}$ in X contains a convergent subsequence.

- Theorem
  (Borel – Lebesgue) Let (X, ρ) *be a metric space. The following are equivalent*

- Theorem
  (Borel – Lebesgue) Let (X, ρ) be a metric space. The following are equivalent
  1. *X is compact.*

- Theorem
  (Borel – Lebesgue) Let (X, ρ) be a metric space. The following are equivalent
  1. *X is compact.*
  2. *X has the Bolzano – Weierstrass Property.*

- **Theorem**
  (Borel – Lebesgue) Let (X, ρ) be a metric space. The following are equivalent
  1. *X is compact.*
  2. *X has the Bolzano – Weierstrass Property.*
  3. *X is sequentially compact.*

- Theorem

  (Borel – Lebesgue) Let (X, ρ) be a metric space. The following are equivalent

  1. *X is compact.*
  2. *X has the Bolzano – Weierstrass Property.*
  3. *X is sequentially compact.*
  4. *X is complete and totally bounded.*