Closed Subsets of Complete Metric Spaces are Complete

Let \((X, \rho)\) be a complete metric space.

\(X\) is called complete if every Cauchy sequence in \(X\) converges to an element \(x \in X\).

Let \((Y, \rho)\) be a metric subspace of \((X, \rho)\).

\(Y\) is complete if and only if \(Y\) is closed in \(X\).

If \(Y\) is complete and \(x\) is a point of closure for \(Y\) then there exists a sequence \(\{y_n\}\) in \(Y\) such that \(y_n \to x\) in \(X\). But then \(\{y_n\}\) is Cauchy in \(Y\) and therefore it must converge to an element of \(Y\). It follows that \(x\) is an element of \(Y\).

Therefore, \(Y\) is closed.

Let \(Y\) be closed in the complete metric space \(X\). Let \(\{y_n\}\) be a Cauchy sequence in \(Y\). Since \(\{y_n\}\) is Cauchy in \(X\) also, then it converges to an element \(x \in X\). But since \(Y\) is closed \(x\) is in \(Y\).

Therefore, \(Y\) is complete.
Closed Subsets of Complete Metric Spaces are Complete

\((X, \rho)\) – Complete Metric Space.
Closed Subsets of Complete Metric Spaces are Complete

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Let $Y$ be closed in the complete metric space $X$. Let $\{y_n\}$ be a Cauchy sequence in $Y$. Since $\{y_n\}$ is Cauchy in $X$ also, then it converges to an element $x \in X$. But since $Y$ is closed $x$ is in $Y$. Therefore, $Y$ is complete.
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\((Y, \rho)\) metric subspace of \((X, \rho)\).
Closed Subsets of Complete Metric Spaces are Complete

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(X, ρ) – Complete Metric Space.

X is called complete if every Cauchy sequence in X converges to an element x ∈ X.

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Y is complete if and only if Y is closed in X.

If Y is complete and x is a point of closure for Y then there exists a sequence \{y_n\} in Y such that \(y_n \to x\) in X. But then \{y_n\} is Cauchy in Y and therefore it must converge to an element of Y. It follows that x is an element of Y. Therefore, Y is closed.
Closed Subsets of Complete Metric Spaces are Complete

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Cantor’s Intersection Theorem

Let \((X, \rho)\) be a complete metric space. For \(E \subseteq X\), the diameter of \(E\) is defined as

\[
\text{diam}(E) = \sup \left\{ \rho(x, y) \mid x, y \in E \right\}
\]

A sequence \(\{E_n\}\) of sets is called descending if \(E_{n+1} \subseteq E_n\) for all \(n\).

**Theorem (Cantor’s Intersection Theorem)**

Let \(\{E_n\}\) be a descending sequence of nonempty closed subsets of the complete metric space \(X\), such that

\[
\lim_{n \to \infty} \text{diam}(E_n) = 0
\]

Then there exists a point \(x \in X\) such that

\[
\bigcap_{n=1}^{\infty} E_n = \{x\}
\]

Choose an element \(x_k\) from each \(E_k\). For every \(n\), the sequence \(\{x_k\}_{k \geq n}\) is a Cauchy sequence in the closed set \(E_n\). So \(\{x_k\}\) converges to an element \(x \in E_n\). \(x\) is the only element contained in all the \(E_n\).
(X, ρ) – Complete Metric Space.
(X, ρ) – Complete Metric Space.

E ⊆ X. The diameter of E is the number
\( \text{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\} \).
Cantor’s Intersection Theorem

Let \( (X, \rho) \) be a complete metric space and \( E \subseteq X \). The diameter of \( E \) is the number
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\text{diam}(E) = \sup \{ \rho(x, y) \mid x, y \in E \}.
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A sequence \( \{E_n\} \) of sets is called \textit{descending} if \( E_{n+1} \subseteq E_n \) for all \( n \).

\[\text{Theorem (Cantor’s Intersection Theorem)}\]
Let \( \{E_n\} \) be a descending sequence of nonempty closed subsets of the complete metric space \( X \), such that
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Then there exists a point \( x \in X \) such that \( \bigcap_{n=1}^{\infty} E_n = \{x\} \).
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E ⊆ X. The diameter of E is the number
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**Theorem**

*(Cantor’s Intersection Theorem)* Let \( \{E_n\} \) be a descending sequence of nonempty closed subsets of the complete metric space \( X \), such that \( \lim_{n \to \infty} \text{diam}(E_n) = 0 \). Then there exists a point \( x \in X \) such that \( \bigcap_{n=1}^{\infty} E_n = \{x\} \).
Real Analysis II

Cantor’s Intersection Theorem

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**Theorem**

\textit{(Cantor’s Intersection Theorem)} Let \(\{E_n\}\) be a descending sequence of nonempty closed subsets of the complete metric space \(X\), such that \(\lim_{n \to \infty} \text{diam}(E_n) = 0\). Then there exists a point \(x \in X\) such that \(\bigcap_{n=1}^{\infty} E_n = \{x\}\).

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Real Analysis II
Some Counterexamples

Let \( a_n = 0.99 \ldots 99 \) and \( b_n = 1 - \frac{1}{10^n} \), and then \( \bigcap_{n=1}^{\infty} (a_n, b_n) = \emptyset \).

\( \bigcap_{n=1}^{\infty} [a_n, \infty) = \emptyset \).
Let \( a_n = 0.99\ldots 99 = 1 - 10^{-n} \), and \( b_n = 1 \). Then
\[
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Let $a_n = 0.99\ldots99 = 1 - 10^{-n}$, and $b_n = 1$. Then

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$$\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset.$$
Open Coverings. Compact Metric Spaces

▶ $(X, \rho)$ – metric space.
▶ $K$ a subset of $X$.
▶ \{O_i\}_{i \in I}$ a family of open subsets of $X$ is called an open covering of $K$ if $K \subseteq \bigcup_{i \in I} O_i$.
▶ $K$ is called compact if every open covering of $K$ has a finite subcovering.
▶ The closed intervals $[a, b]$ are compact subsets of $\mathbb{R}$.
▶ The open intervals $(a, b)$, with $a < b$ are not compact subsets of $\mathbb{R}$.
▶ Closed and bounded subsets of $\mathbb{R}^n$ are compact subsets of $\mathbb{R}^n$.
▶ The closed ball $B(0, 1) \subseteq L^p$ is not compact for any $p \in [1, \infty]$.
(X, ρ) – metric space. K a subset of X.
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\[
\{O_i\}_{i \in I} \text{ a family of open subsets of } X \text{ is called an open covering of } K \text{ if } K \subseteq \bigcup_{i \in I} O_i.
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\(K\) is called **compact** if every open covering of \(K\) has a finite subcovering.

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- The closed ball \(B(0, 1) \subseteq L^p\) is not compact for any \(p \in [1, \infty]\).
Let $F$ be a collection of subsets of a metric space $X$. We say that $F$ has the finite intersection property if the intersection of any finite number of sets from $F$ is nonempty.

$F$ has the finite intersection property if and only if for any finite subcollection $\{F_1, \ldots, F_n\}$ of sets from $F$, $\bigcap_{i=1}^{n} F_i \neq X$.

A metric space $X$ is compact if and only if every collection $F$ of closed subsets of $X$ that has the finite intersection property has a nonempty intersection.

$(\Rightarrow)$ Let $X$ be compact and $F$ be a collection of closed subsets of $X$ with the finite intersection property. Let $A = \bigcap\{F \mid F \in F\}$. $A' = \bigcup_{F \in F} F'$ form an open covering of $A'$. Since no finite subcollection $\{F' \mid F \in F\}$ covers $X$, and $X$ is compact, then $A' \neq X$. Therefore, $A \neq \emptyset$. 

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$(\Rightarrow)$ Let $X$ be compact and $\mathcal{F}$ be a collection of closed subsets of $X$ with the finite intersection property. Let $A = \bigcap\{F_{\mid \mid \mid} F \in \mathcal{F}\}$. $A' = \bigcup F \in \mathcal{F} F'$. The $F'$ form an open covering of $A'$. Since no finite subcollection $\{F'_1, \ldots, F'_n\}$ covers $X$, and $X$ is compact, then $A' \neq X$. Therefore, $A \neq \emptyset$. 

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Let $X$ be a metric space such that every collection of closed sets that has the finite intersection property has a nonempty intersection. Let $\{O_i\}_{i \in I}$ be any open covering of $X$. Assume that no finite subcollection of $\{O_i\}_{i \in I}$ covers the whole space $X$. Then the family $F = \{O'_i\}_{i \in I}$ has the finite intersection property. Consequently, $\emptyset = (\bigcup_{i \in I} O_i)' = \bigcap_{i \in I} O'_i \neq \emptyset$. Contradiction.
(⇐) Let $X$ be a metric space such that every collection of closed set that has the finite intersection property has a nonempty intersection. Let $\{O_i\}_{i \in I}$ be any open covering of $X$. Assume that no finite subcollection of $\{O_i\}_{i \in I}$ covers the whole space $X$. Then the family $F = \{O'_i\}_{i \in I}$ has the finite intersection property. Consequently, $\emptyset = (\bigcup_{i \in I} O_i)' = \bigcap_{i \in I} O'_i \neq \emptyset$. Contradiction.
Totally bounded sets

$(X, \rho)$ – a metric space.

$K$ a subset of $X$.

We say that $K$ is totally bounded (subset of $X$) if for every $\varepsilon > 0$ there exists a finite set of balls of radius $\varepsilon$ all centered in $K$ such that $K$ is contained in the union of those balls.

$K$ is called bounded if there exists a ball $B(x, r)$ in $X$ such that $K \subseteq B(x, r)$.

Totally bounded implies bounded.

The ball $B(0, 1)$ in $L^2$ is bounded but not totally bounded.
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Bolzano – Weierstrass Property. Sequential Compactness

Let $(X, \rho)$ be a metric space.

$X$ is said to have the Bolzano – Weierstrass Property if every infinite sequence has a cluster point.

$x \in X$ is a cluster point for $\{x_n\}$ if every open neighborhood of $x$ contains infinitely many terms of the sequence.

$X$ is called sequentially compact if every sequence $\{x_n\}$ in $X$ contains a convergent subsequence.
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Theorem (Borel – Lebesgue) Let \((X, \rho)\) be a metric space. The following are equivalent:

1. \(X\) is compact.
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