

Real Analysis II

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- Every normed linear space $(X, \|\cdot\|)$ is a metric space with the metric $\rho(x, y) = \|x - y\|$.
- Let (X, ρ) be a metric space, $x \in X$, and r be a positive number. The (open) ball of radius r centered at x is the set $B(x, r) = \{y \in X \mid \rho(x, y) < r\}$.

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Open Sets in Metric Spaces

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- ▶ \mathcal{O} is called a (open) neighborhood of x if \mathcal{O} is open and contains x .

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- ▶ A set E is closed if and only if $E' = X \setminus E$ is open.

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Convergence and Completeness

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- ▶ x is a point of closure for a set $E \subseteq X$ if and only there is a sequence in E that converges to x in X .

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- ▶ f is said to be continuous at $x \in X$ if for every $\varepsilon > 0$ there exists a ball $B(x, r) \subseteq X$ such that $f(B(x, r)) \subseteq B(f(x), \varepsilon)$.

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f is continuous if and only if for every open subset \mathcal{O} of Y its preimage $= f^{-1}(\mathcal{O})$ is an open subset of X .