Let $E$ be a measurable set. For $p \in (0, \infty)$ let $L^p = L^p(E)$ be the space of measurable functions $f$ such that $\int_E |f|^p < \infty$.

$L^\infty$ is the space of measurable functions on $E$ that are essentially bounded.

Two functions are considered the same as elements of $L^p$ if they differ only on a set of measure zero.

If $1 \leq p < \infty$, then $L^p$ is a normed linear space with the norm $\|f\|_p = \left(\int |f|^p\right)^{1/p}$ for $1 \leq p < \infty$ and $\|f\|_\infty = \text{ess sup} |f(x)|$.

$L^p$ is complete for all $p \in [1, \infty]$. This means that every Cauchy sequence in $L^p$ converges to a function in $L^p$.

This makes $L^p$ to be a Banach Space.

Banach Space = Complete Normed Linear Space.
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This makes $L^p$ to be a Banach Space.

Banach Space $=$ Complete Normed Linear Space.
A sequence \( \{f_n\} \) of functions in \( L^p \) that converges in \( L^p \) to a function \( f \in L^p \) need not converge pointwise a.e. to \( f \).

If the sequence \( \{f_n\} \) converges to \( f \) in \( L^p \) then there is a subsequence \( \{f_{n_k}\} \) that converges to \( f \) pointwise almost everywhere.

A sequence \( \{f_n\} \) of functions in \( L^p \) that converges pointwise a.e. to a function \( f \in L^p \) need not converge to \( f \) in \( L^p \).

Theorem
Let \( \{f_n\} \) be a sequence of functions in \( L^p(E) \) that converges pointwise a.e. to a function \( f \in L^p \).

Then \( \{f_n\} \) converges to \( f \) (in norm) in \( L^p \) if and only if
\[
\lim_{n \to \infty} ||f_n||_p = ||f||_p.
\]
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Proof.
For case $1 \leq p < \infty$. 

$\left| f_n - f \right|^p \leq \left| f_n \right|^p + \left| f \right|^p$ 

Let $h_n = \left| f_n \right|^p + \left| f \right|^p - \left| f_n - f \right|^p$ 

Then $h_n \geq 0$, $h_n \to \left| f \right|^p$ pointwise a.e. and $\int h_n \leq M$ ($M = 4(\left| f \right|^p + 1)$ does the job (Fatou) 

$\left| f \right|^p \leq \liminf \int h_n \leq \left| f \right|^p - \limsup \int \left| f_n - f \right|^p$ 

Therefore, $f_n \to f$ in $L^p$. 

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Proof.
For case $1 \leq p < \infty$.

- Let $f_n \rightarrow f$ pointwise a.e. and $\|f_n\|_p \rightarrow \|f\|_p$. 
Real Analysis II

Pointwise Convergence + Convergence of norms = Convergence in Norm

Proof.
For case $1 \leq p < \infty$.

- Let $f_n \to f$ pointwise a.e. and $\|f_n\|_p \to \|f\|_p$.
- $|t|^p$ is convex
Proof.

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- Let $f_n \rightarrow f$ pointwise a.e. and $\|f_n\|_p \rightarrow \|f\|_p$.
- $|t|^p$ is convex
- $\left| \frac{f_n - f}{2} \right|^p \leq \frac{|f_n|^p + |f|^p}{2}$
Proof.
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- Let $h_n = \frac{|f_n|^p + |f|^p}{2} - \left| \frac{f_n - f}{2} \right|^p$
- Then $h_n \geq 0$, $h_n \to |f|^p$ pointwise a.e. and $\int h_n \leq M$ ($M = 4(\|f\|_p^p + 1)$) does the job.
Proof.
For case $1 \leq p < \infty$.

1. Let $f_n \rightarrow f$ pointwise a.e. and $\|f_n\|_p \rightarrow \|f\|_p$.
2. $|t|^p$ is convex.
3. $\left| \frac{f_n-f}{2} \right|^p \leq \frac{|f_n|^p + |f|^p}{2}$
4. Let $h_n = \frac{|f_n|^p + |f|^p}{2} - \left| \frac{f_n-f}{2} \right|^p$
5. Then $h_n \geq 0$, $h_n \rightarrow |f|^p$ pointwise a.e. and $\int h_n \leq M$ ($M = 4(\|f\|_p^p + 1)$) does the job.
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Proof.
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- Let $f_n \rightarrow f$ pointwise a.e. and $\|f_n\|_p \rightarrow \|f\|_p$.
- $|t|^p$ is convex
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- Then $h_n \geq 0$, $h_n \rightarrow |f|^p$ pointwise a.e. and $\int h_n \leq M$ ($M = 4(\|f\|_p^P + 1)$) does the job
- (Fatou) $\|f\|_p^P \leq \lim\inf \int h_n \leq \|f\|_p^P - \lim\sup \int \left| \frac{f_n - f}{2} \right|^p$
- Therefore, $f_n \rightarrow f$ in $L^p$. 

$\Box$
The functions in $L^p$ need not be continuous. But how bad can they be?

Functions in $L^p$ can be approximated by continuous functions if $1 \leq p < \infty$.

**Theorem**

Let $1 \leq p < \infty$, $E$ – a measurable set, $f$ any function in $L^p(E)$, and $\varepsilon$ any positive number. Then there exists a continuous function $\phi$ defined on $E$ such that $||f - \phi||_p < \varepsilon$.

Functions with jump discontinuities cannot be approximated in $L_\infty$ by continuous functions.
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Functions with jump discontinuities cannot be approximated in $L^\infty$ by continuous functions.
Definition

Let \((X, \| \cdot \|)\) be a normed linear space. Let \(A \subseteq B \subseteq X\). We say that \(A\) is dense in \(B\) if for every \(b \in B\) and for every \(\varepsilon > 0\) there exists an element \(a \in A\) such that \(\|a - b\| < \varepsilon\).
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\(C(E) = \) the set of continuous functions defined on \(E\) is dense in \(L^p\) for \(1 \leq p < \infty\), but it is not dense in \(L^\infty(E)\).
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\[ C(E) = \text{the set of continuous functions defined on } E \text{ is dense in } L^p \text{ for } 1 \leq p < \infty, \text{ but it is not dense in } L^\infty(E). \]

Theorem

Let \(E\) be a measurable set, \(S(E)\) be the space of simple functions defined on \(E\), and \(1 \leq p \leq \infty\). Then \(S(E)\) is dense in \(L^p(E)\).
Separability of $L^p$ for $1 \leq p < \infty$

A normed linear space $X$ is called separable if there exists a countable subset $A$ of $X$ that is dense in $X$.

If $1 \leq p < \infty$ then $L^p$ is separable. $L^\infty$ is not separable.

The family of functions $F = \{ \phi_s = \chi_{[0,s]} \}$ $s \in \mathbb{R}$ is uncountable.

If $s \neq t$ then $||\phi_s - \phi_t||_\infty = 1$.

Therefore, no countable subset $A$ of $L^\infty$ can satisfy that for each element $f$ of $F$ there is an element $a \in A$ such that $||a - f||_\infty < 1/2$. 
Definition

A normed linear space $X$ is called *separable* if there exists a *countable* subset $A$ of $X$ that is dense in $X$. 

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Let $V, W$ be two linear spaces. A mapping $T: V \to W$ is called linear if it sends linear combinations to linear combinations, i.e., if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in V$ and all scalars $\alpha, \beta$.

Let $X$ be two normed linear spaces and $T: X \to \mathbb{R}$ a linear mapping. $T$ is called a linear bounded functional on $X$ if there is a real number $M$ such that $||T(x)|| \leq M||x||$ for all $x \in X$.

Here $||T(x)||$ is simply the absolute value of $T(x)$. 
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Definition

Let $X$ be a normed linear space and $T$ a linear bounded functional on $X$. The norm of $T$ is the infimum of all numbers $M$ such that $\|T(x)\| \leq M \|x\|$. The norm of $T$ is denoted by $\|T\|$.

The set of all linear bounded functionals on a normed linear space $X$ is a linear space.

Definition

Let $X$ be a normed linear space. The space of all linear bounded functionals on $X$ is called the dual space of $X$. It is denoted by $X^*$.

$(X^*, \|\cdot\|)$ is a normed linear space.
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- $(X^*, \| \cdot \|)$ is a normed linear space.
Theorem

Let $1 \leq p < \infty$ and $q = \frac{p}{p-1}$ its conjugate. For every function $g \in L^q$ the mapping $T: f \in L^p \rightarrow T(f) = \int g f \in \mathbb{R}$ is a bounded linear functional on $L^p$.

Theorem (Riesz Representation Theorem) Let $1 \leq p < \infty$ and $q = \frac{p}{p-1}$ its conjugate. Let $T$ be a bounded linear functional on $L^p$. Then there exists a function $g \in L^q$ such that $T(f) = \int g f$ for every function $f \in L^p$.

For $T$, $g$ in this theorem, $\|T\| = \|g\|_q$.

So for $1 \leq p < \infty$, the dual space of $L^p$ is (isometric to) $L^q$. 

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Theorem

Let $1 \leq p < \infty$ and $q = \frac{p}{p-1}$ its conjugate. For every function $g \in L^q$ the mapping $T : f \in L^p \longrightarrow T(f) = \int g f \in \mathbb{R}$ is a bounded linear functional on $L^p$. 

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- For $T, g$ in this theorem, $\|T\| = \|g\|_q$.
- So for $1 \leq p < \infty$, the dual space of $L^p$ is (isometric to) $L^q$. 