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- Theorem

  Let $\{f_n\}$ be a sequence of functions in $L^p(E)$ that converges pointwise a.e. to a function $f \in L^p$. Then $\{f_n\}$ converges to f (in norm) in L^p if and only if $\lim_{n\to\infty} ||f_n||_p = ||f||_p$.

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- Therefore, $f_n \to f$ in L^p.

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  Let E be a measurable set, $S(E)$ be the space of simple functions defined on E, and $1 \leq p \leq \infty$. Then $S(E)$ is dense in $L^p(E)$.

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  - Therefore, no countable subset A of L^∞ can satisfy that for each element f of $\mathcal{F}$ there is an element $a \in A$ such that $||a - f||_\infty < 1/2$.

- Let V, W be two linear spaces. A mapping $T : V \longrightarrow W$ is called *linear* if it sends linear combinations to linear combinations, i.e., if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in V$ and all scalars α, β.

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  - For T, g in this theorem, $||T|| = ||g||_q$.
  - So for $1 \leq p < \infty$, the dual space of L^p is (isometric to) L^q.