MATH 135 Lecture Notes | Sections 7.5 & 7.6
November 5, 2006 | Inverse Trigonometric Functions

The inverse trigonometric functions give the value of an angle (usually in radians) for which the indicated trigonometric function takes on a certain value.

For example: $\sin\frac{\pi}{6} = \frac{1}{2}$, and $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ (read "inverse sine of one half equals pi over six").

From this example, note three important things about inverse trigonometric functions.

1. Caution (on page 247): the -1 in $\sin^{-1}$ is **not an exponent**! $\sin^{-1} x \neq \frac{1}{\sin x} = \csc x$
2. The inverse trig functions on a calculator are found by 2nd sin 2nd cos, and 2nd tan to give $\sin^{-1}$, etc.

   On a calculator $\sin^{-1} .5 = 0.523598776$, a decimal approximation for $\frac{\pi}{6}$.
3. Since $\frac{1}{2} = \sin\frac{\pi}{6} = \sin\frac{5\pi}{6} = \sin\frac{13\pi}{6}$ etc., the range of the inverse sine function must be **restricted** so that for every value of x there is a unique value of $\sin^{-1} x$.
   The ranges of the six inverse trigonometric functions are given in the box on page 248.

Graphs of inverse trigonometric functions

Study Figures 7-41, 7-42, and 7-43 at the bottom of page 249. Notice how these graphs were obtained first by sketching $x = \sin y$, $x = \cos y$, and $x = \tan y$ and then restricting the range to the values of y indicated on page 248. The heavier, colored portions indicate the graphs of the inverse trigonometric functions $y = \sin^{-1} x$, $y = \cos^{-1} x$, and $y = \tan^{-1} x$.

You should be able explain why the domain of the inverse sine and inverse cosine functions is $\{x | -1 \leq x \leq 1\}$ whereas the domain of the inverse tangent function is the set of real numbers.

We are now ready to find the derivatives of the inverse trigonometric functions.
We use the fact if one function is the inverse of another, the derivative of the first function is the reciprocal of the derivative of the second function. This point is illustrated for the inverse sine function. (Formula 7-40 on page 253).

Let $y = \sin^{-1} x$. Then $x = \sin y$.

$\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$ (Note we have used a Pythagorean Identity here.)

Therefore $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$, so long as the denominator is not zero, i.e. for $-1 < x < 1$.

Take a close look at the graph in figure 7-41 on page 249 to see why $\frac{dy}{dx}$ is not defined for $x = -1$ and $x = 1$.

Similarly the equations 7-41 and 7-42 give the derivatives for the inverse cosine and inverse tangent functions.

Again look at the graphs on page 249 to see why the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ are opposites of each other.

Three examples in these sections show some applications for the inverse trigonometric functions.

Example 7, page 250. Note how the double angle formula is used to rewrite $\sin \omega t \cos \omega t$ as $\frac{1}{2} \sin 2\omega t$.

Example 2, page 254, shows how an inverse trigonometric function is used in a related rates problem.

Example 6, page 255, shows calculation of the slope of a tangent line. Note the use of the quotient rule.

Exercises:

page 252-253: 9, 11, 13, 17, 29, 45, 47, 61, 67, 71
page 256: 1, 3, 7, 9, 13, 17, 41, 43, 47