

Example 4 page 268 shows a table and graph for the exponential function  $y = f(x) = 2^x$ .

Note these characteristics:

- (1) the domain is all real numbers; the range is all positive numbers
- (2) the negative x-axis is an asymptote
- (3) the first derivative is positive everywhere and increasing
- (4) the second derivative is positive everywhere; the curve is concave up everywhere.

Based on these characteristics we can predict that the graph of the first derivative,  $y = f'(x)$  should look a lot like the original function. We can test that prediction using the nDeriv function on the calculator. In the Y = menu enter:

$$Y1 = 2^X$$

$$Y2 = \text{nDeriv}(Y1, X, X)$$

To find nDeriv: MATH 8

To find Y1: VARS scroll right to YVARS 1, 1

Plotting Y1 and Y2 in the ZDecimal window we should notice that the graph of Y2 lies below the graph of Y1.

Tracing gives y-intercepts (0,1) for Y1 and approximately (0,0.693) for Y2.

Now let's examine another exponential function  $y = f(x) = 4^x$ . Change to  $Y1 = 4^X$  and press graph. Notice that the graph of  $y = 4^x$  is steeper than the graph of  $y = 2^x$  and that the graph of its derivative lies above the graph of the function. Tracing gives y-intercepts (0,1) for Y1 and approximately (0,1.386) for Y2. Furthermore, we can use the TABLE feature to show the ratio  $Y2/Y1$  appears to be the same for all values of X, approximately 1.386.

Conjecture: If  $y = b^x$  for some positive number  $b$ , then  $\frac{dy}{dx} = kb^x$  where  $k$  is a constant that depends upon  $b$ . (When  $b = 2$ , it appears that  $k \approx 0.693$  and when  $b = 4$ ,  $k \approx 1.386$ .)

Here is a proof of the conjecture using the "delta process" based on the definition of derivative:

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{b^{x+\Delta x} - b^x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{b^x b^{\Delta x} - b^x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left( \frac{b^{\Delta x} - 1}{\Delta x} \right) b^x$$

The constant  $k$  is then  $\lim_{\Delta x \rightarrow 0} \frac{b^{\Delta x} - 1}{\Delta x}$ , provided that the limit exists. You may check this limit for

$b = 2$  and  $b = 4$  by taking values of  $\Delta x$  close to 0. Then use the calculator to show that you get the same result by taking the **natural logarithm** of  $b$ . ( $\ln 2 \approx 0.693$  and  $\ln 4 \approx 1.386$ .)

Conclusion:  $\frac{d(b^x)}{dx} = (\ln b)b^x$ . Compare with equation 8-13 on page 276.

Recall that the number  $e \approx 2.718$  is the base of natural logarithms and that  $\ln e = 1$ .

It therefore follows that  $\frac{d(e^x)}{dx} = e^x$ . Compare with equation 8-14 on page 276.

We can see the derivative of the natural exponential function by graphing.

$$Y1 = e^X$$

$$Y2 = \text{nDeriv}(Y1, X, X)$$

and observing that the graphs of Y1 and Y2 are the same. One way of describing the number  $e$  is to say that  $f(x) = e^x$  is the unique function that is equal everywhere to its first derivative.

Review the relationship between logarithms and exponents by studying Figure 8-1 and Examples 1, 2, and 3 on page 267.

Recall that there are two special bases for logarithms. The notation **log x** is used for base 10 or **common** logarithms. The notation **ln x** is used for base  $e$  or **natural** logarithms. Both log and ln appear on the calculator. To find logarithms in other bases you may use the change of base formula (8-10) near the bottom on page 270.

Finally consider what happens when  $0 < b < 1$ . For example,  $b = \frac{1}{2}$ .  $\ln(0.5) \approx -0.693$ .

The graph of  $y = f(x) = \left(\frac{1}{2}\right)^x$  is an example of exponential “decay” rather than “growth”. Its derivative is negative everywhere. The graph of the derivative lies in the 3<sup>rd</sup> and 4<sup>th</sup> quadrants.

Also note that this function may be written  $y = f(x) = 2^{-x}$ . Use the chain rule to show that the derivative is the same.

Exercises:

page 271: 1, 3, 5, 9, 13, 15 (without a calculator), 35, 41, 45, 47

page 278-9: 1, 4, 5, 9, 15, 47, 49

Find the fifth derivative of  $f(x) = e^x$ .