CARDINALITIES OF WEAKLY LINDELÖF SPACES WITH REGULAR $G_\kappa$-DIAGONALS

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ABSTRACT. For a Urysohn space $X$ we define the regular diagonal degree $\Delta(X)$ of $X$ to be the minimal infinite cardinal $\kappa$ such that $X$ has a regular $G_\kappa$-diagonal i.e. there is a family $(U_\eta : \eta < \kappa)$ of open sets in $X^2$ such that $\{(x,x) : x \in X\} = \bigcap_{\eta < \kappa} U_\eta$.

In this paper we show that if $X$ is a Urysohn space then: (1) $|X| \leq 2^{wL(X)} \Delta(X)$; (2) $|X| \leq wL(X)^{\Delta(X)} \chi(X)$; and (3) $|X| \leq aL(X)^{\Delta(X)}$; where $\chi(X)$, $wL(X)$ and $aL(X)$ are respectively the character, the weak Lindelöf number and the almost Lindelöf number of $X$.

It follows from (1) that the cardinality of every Urysohn space $X$ does not exceed $2^{\chi(X)} \Delta(X)$ and that every weakly Lindelöf space with a regular $G_\delta$-diagonal has cardinality at most $2^{wL}$.

Inequality (2) implies that when $X$ is a space with a regular $G_\delta$-diagonal then $|X| \leq wL(X)^{\chi(X)}$. This improves significantly Bell, Ginsburg and Woods inequality $|X| \leq 2^{\chi(X)wL(X)}$, which is known to be true for normal $T_1$-spaces, is true also for a large class of Urysohn spaces that includes all spaces with regular $G_\delta$-diagonals.

For the class of spaces with regular $G_\delta$-diagonals (3) improves Bella and Cammaroto inequality $|X| \leq 2^{\chi(X)aL(X)}$, which is valid for all Urysohn spaces. Also, it follows from (3) that the cardinality of every space with a regular $G_\delta$-diagonal does not exceed $aL(X)^{\omega}$.

1. Introduction

Perhaps the two most famous results involving cardinal functions are Arhangel’skii’s and Hajnal-Juhás’s theorems asserting that if $X$ is a Hausdorff space then $|X| \leq 2^{\chi(X)L(X)}$ [1] and $|X| \leq 2^{\chi(X)c(X)}$ [19], where $\chi(X)$, $L(X)$ and $c(X)$ denote respectively the character, Lindelöf number and cellularity of $X$.

Bell, Ginsburg and Woods showed in [4] that if $X$ is a normal $T_1$-space then

\begin{equation}
|X| \leq 2^{\chi(X)wL(X)}
\end{equation}
where $wL(X)$ is the weak Lindelöf number of $X$. Since $wL(X) \leq L(X)$ and $wL(X) \leq c(X)$, (1) generalizes (for the class of normal spaces) Arhangel’skii’s and Hajnal-Juhás’ inequalities. In the same paper the authors constructed an example (see [4, Example 2.3]) showing that for Hausdorff spaces the gap between $|X|$ and $2^{\chi(X)wL(X)}$ could be arbitrarily large and they asked (see [4, 4.1]) if (1) holds true for all regular $T_1$-spaces. To the best of our knowledge this question is still open (see [17, Question 1]). In this paper we give a partial answer to their question by showing that for every space $X$ with a regular $G_\delta$-diagonal even the stronger inequality $|X| \leq wL(X)^{\chi(X)}$ is true.

In 1977, Ginsburg and Woods proved (see [12]) that if $X$ is a $T_1$-space then $|X| \leq 2^{e(X)\Delta(X)}$, where $e(X)$ and $\Delta(X)$ denote respectively the extent and the diagonal degree of $X$. As a corollary of that inequality the authors obtained that if $X$ is a collectionwise Hausdorff space then

$$|X| \leq 2^{e(X)\Delta(X)}. \tag{2}$$

They also noticed (see [12, Example 2.4]) that the Katětov extension $k\omega$ of the countable discrete space $\omega$ is an example of a Urysohn space (every two points have disjoint closed neighborhoods) for which $|k\omega| > 2^{e(k\omega)\Delta(k\omega)}$ and they asked if (2) was true for every regular $T_1$-space [12, Question 2.5]. In 1978 Arhangel’skii independently asked the countable version of that same question: Is it true that if $X$ is a regular ccc-space with a $G_\delta$-diagonal then $|X| \leq 2^\omega$ (see [2, p. 91, Question 16]). Shakhmatov answered their question in [20] by showing that there is no upper bound for the cardinality of completely regular ccc-spaces with $G_\delta$-diagonals. Then Arhangel’skii asked what if “$G_\delta$-diagonal” is replaced by “regular $G_\delta$-diagonal” [8]. Buzyakova answered that question by proving the following theorem:

**Theorem 1.1** ([8]). The cardinality of a ccc-space with a regular $G_\delta$-diagonal does not exceed $2^\omega$.

In this paper we show that if $X$ is a Urysohn space then:

$$|X| \leq 2^{wL(X)\Delta(X)} \tag{3}$$

$$|X| \leq wL(X)^{\Delta(X)\chi(X)} \tag{4}$$

$$|X| \leq aL(X)^{\Delta(X)} \tag{5}$$

It follows from (3) that the cardinality of every Urysohn space does not exceed $2^{wL(X)\Delta(X)}$ and that every weakly Lindelöf space with regular $G_\delta$-diagonal has cardinality at most $2^\omega$. This generalizes Theorem 1.1. It also
trivially follows from (3) that $|X| \leq 2^{\chi(X) \cdot w_L(X) \cdot \Delta(X)}$; hence Bell, Ginsburg
and Woods inequality $|X| \leq 2^{\chi(X) \cdot w_L(X)}$ is true for all Urysohn spaces for
which $\chi(X) \cdot w_L(X) \geq \Delta(X)$. This class of spaces clearly includes all spaces
with regular $G_\delta$-diagonals.

Inequality (4) implies that when $X$ is a space with a regular $G_\delta$-diagonal
then $|X| \leq w_L(X)^{\chi(X)}$. This improves significantly Bell, Ginsburg and
Woods inequality for the class of normal spaces with regular $G_\delta$-diagonals.
In particular (4) shows that the cardinality of every first countable space
with regular $G_\delta$-diagonal does not exceed $w_L(X)^\omega$.

For the class of spaces with regular $G_\delta$-diagonals (5) improves Bella and
Cammaroto inequality $|X| \leq 2^{\chi(X) \cdot a_L(X)}$, which is valid for all Urysohn
spaces [5]. Also, it follows from (5) that the cardinality of every space with
a regular $G_\delta$-diagonal does not exceed $a_L(X)^\omega$.

2. Definitions

Throughout this paper $\omega$ is (the cardinality of) the set of all non-negative
integers, $\xi$ and $\eta$ are ordinals and $\tau$, $\mu$ and $\kappa$ are infinite cardinals. The
cardinality of the set $X$ is denoted by $|X|$ and $\Delta_X := \{(x, x) \in X^2 : x \in X\}$
is the diagonal of $X$. If $\mathcal{U}$ is a family of subsets of $X$, $x \in X$, and $G \subset X$ then
$\text{st}(G, \mathcal{U}) := \bigcup\{U \in \mathcal{U} : U \cap G \neq \emptyset\}$. When $G = \{x\}$ we write $\text{st}(x, \mathcal{U})$ instead
of $\text{st}(\{x\}, \mathcal{U})$. If $n \in \omega$, $\text{st}^n(G, \mathcal{U}) = \text{st}(\text{st}^{n-1}(G, \mathcal{U}), \mathcal{U})$ and $\text{st}^0(G, \mathcal{U}) = G$.

All spaces are assumed to be topological $T_1$-spaces. For a subset $U$ of a
space $X$ the closure of $U$ (in $X$) is denoted by $\overline{U}$. $F \subset X$ is called regular-
closed (in $X$) if there is open $U \subset X$ such that $F = \overline{U}$. As usual, $\chi(X)$ and
$\psi(X)$ denote respectively the character and the pseudocharacter of $X$. The
closed pseudocharacter $\psi_c(X)$ (defined only for Hausdorff spaces $X$) is the
smallest infinite cardinal $\kappa$ such that for each $x \in X$, there is a collection
$\{V(\eta, x) : \eta < \kappa\}$ of open neighborhoods of $x$ such that $\cap_{\eta<\kappa} V(\eta, x) = \{x\}$
[21]. The Hausdorff pseudocharacter of $X$, denoted $H\psi(X)$, is the smallest
infinite cardinal $\kappa$ such that for each $x \in X$, there is a collection $\{V(\eta, x) : \eta < \kappa\}$ of open neighborhoods of $x$ such that if $x \neq y$, then there exists $\eta, \xi < \kappa$ such that $V(\eta, x) \cap V(\xi, y) = \emptyset$ [17].

The Lindelöff number of $X$ is $L(X) := \min\{\kappa : \text{every open cover of } X \text{ has a subcover of cardinality } \leq \kappa\} + \omega$. The weak Lindelöff number of $X$, denoted $wL(X)$, is the smallest infinite cardinal $\kappa$ such that every open
cover of $X$ has a subcollection of cardinality $\leq \kappa$ whose union is dense in
$X$. If $wL(X) = \omega$ then $X$ is called weakly Lindelöf. The almost Lindelöff
number of $X$, denoted $aL(X)$, is the smallest infinite cardinal $\kappa$ such that
for every open cover \( U \) of \( X \) there is a subcollection \( U_0 \) such that \( |U_0| \leq \kappa \) and \( \bigcup \{ U : U \in U_0 \} = X \). If \( aL(X) = \omega \) then \( X \) is called almost Lindelöf. 

\( e(X) := \sup\{|D| : D \subseteq X \text{ is closed and discrete}\} + \omega \) is the extent of \( X \). A pairwise disjoint collection of non-empty open sets in \( X \) is called a cellular family. The cellularity of \( X \) is \( c(X) := \sup\{|U| : U \text{ a cellular family in } X\} + \omega \). If \( c(X) = \omega \) then it is called that \( X \) satisfies the countable chain condition (or ccc) property.

A space \( X \) has a \( G_\kappa \)-diagonal if there is a family \( \{U_\eta : \eta < \kappa\} \) of open sets in \( X^2 \) such that \( \Delta_X = \bigcap_{\eta<\kappa}U_\eta \); if \( \Delta_X = \bigcap_{\eta<\kappa}U_\eta \) then it is called that \( X \) has a regular \( G_\kappa \)-diagonal. When \( \kappa = \omega \) then it is called that \( X \) has a \( G_\delta \)-diagonal (respectively, regular \( G_\delta \)-diagonal). The diagonal degree of \( X \), denoted \( \Delta(X) \), is the smallest infinite cardinal \( \kappa \) such that \( X \) has a \( G_\kappa \)-diagonal (hence \( \Delta(X) = \omega \) if and only if \( X \) has a \( G_\delta \)-diagonal).

The following observation is well-known and easy to prove (see e.g. [10, Lemma 4.6]).

**Lemma 2.1.** \( X \) has a diagonal which is the intersection of some of its regular-closed neighborhoods if and only if \( X \) is a Urysohn space.

**Definition 2.2.** For a Urysohn space \( X \) we define the regular diagonal degree \( \overline{\Delta}(X) \) of \( X \) to be the minimal infinite cardinal \( \kappa \) such that \( X \) has a regular \( G_\kappa \)-diagonal.

Let \( n \) be a positive integer. \( X \) has a rank \( n \)-diagonal (a strong rank \( n \)-diagonal) if there is a sequence \( \{U_m : m < \omega\} \) of open covers of \( X \) such that for all \( x \neq y \), there is some \( m < \omega \) such that \( y \notin \text{st}^n(x,U_m) \) (\( y \notin \overline{\text{st}^n(x,U_m)} \) (\cite{3}, \cite{6}). Spaces \( X \) with rank \( n \)-diagonals and strong rank \( n \)-diagonals were introduced and first studied in \cite{18} under the names “spaces with \( G_\delta(n) \)-diagonals” and “spaces with \( \overline{G}_\delta(n) \)-diagonals”. Clearly the spaces with strong rank 1-diagonals or, equivalently, the spaces with \( \overline{G}_\delta(1) \)-diagonals, are exactly the spaces with \( G_\delta^* \)-diagonals introduced and studied in \cite{14}.

The rank (strong rank) of the diagonal of a space \( X \) is defined as the greatest natural number \( n \) such that \( X \) has a rank \( n \)-diagonal (strong rank \( n \)-diagonal), if such a number \( n \) exists. The rank (strong rank) of the diagonal of \( X \) is infinite, if \( X \) has a rank \( n \)-diagonal (strong rank \( n \)-diagonal) for every \( n \geq 1 \) (\cite{3}, \cite{6}).

Condensations are one-to-one and onto continuous mappings. A space \( X \) is submetrizable if it condenses onto a metrizable space \cite{3}, or equivalently, \((X,\tau)\) is submetrizable if there exists a topology \( \tau' \) on \( X \) such that \( \tau' \subset \tau \) and \((X,\tau')\) is metrizable \cite{13}.
A sequence \((U_n : n < \omega)\) of open covers of a space \(X\) is a \textit{development} for \(X\) if for each \(x \in X\), the set \(\{\text{st}(x, U_n) : n < \omega\}\) is a base at \(x\). A \textit{developable space} is a space that has a development. A \textit{Moore space} is a regular developable space [13].

For definitions not given here we refer the reader to [11], [19] or [15].

3. Preliminary results

The following three observations are well-known.

Lemma 3.1. For every Hausdorff space \(X\)
\[\psi(X) \leq \psi_c(X) \leq H\psi(X) \leq \chi(X).\]

Lemma 3.2. \(wL(X) \leq aL(X) \leq L(X).\)

Lemma 3.3 ([15]). \(wL(X) \leq c(X).\)

In 1961 Ceder made the following observation.

Lemma 3.4 ([9]). A space \(X\) has a \(G_\delta\)-diagonal if and only if there is a sequence \((U_n : n < \omega)\) of open covers of \(X\) such that if \(x \in X\), then \(\{x\} = \cap_{n<\omega} \text{st}(x, U_n)\).

It is then clear that if a space \(X\) has a \(G_\delta\)-diagonal then \(\psi(X) = \omega\) and that \(X\) has a \(G_\delta\)-diagonal if and only if \(X\) has a rank 1-diagonal.

In [22] Zenor proved the following lemma.

Lemma 3.5. A space \(X\) has a regular \(G_\delta\)-diagonal if and only if there is a family \((U_\eta : \eta < \kappa)\) of open covers of \(X\) such that if \(x\) and \(y\) are distinct points of \(X\), then there is an integer \(n < \omega\) and open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively, such that no member of \(U_\eta\) intersects both \(U\) and \(V\).

In fact Zenor’s proof of Lemma 3.5 proves a little bit more. For completeness we provide here Zenor’s proof for spaces with regular \(G_\kappa\)-diagonals.

Lemma 3.6. A space \(X\) has a regular \(G_\kappa\)-diagonal if and only if there is a family \((U_\eta : \eta < \kappa)\) of open covers of \(X\) such that if \(x\) and \(y\) are distinct points of \(X\), then there is \(\eta < \kappa\) and open sets \(U \in U_\eta\) and \(V \in U_\eta\) containing \(x\) and \(y\) respectively, such that no member of \(U_\eta\) intersects both \(U\) and \(V\).

Proof. Suppose that \(X\) has a regular \(G_\kappa\)-diagonal. Let \((W_\eta : \eta < \kappa)\) be a family of open sets in \(X^2\) such that \(\Delta_X = \cap_{\eta<\kappa} W_\eta = \cap_{\eta<\kappa} \overline{W_\eta}\). For each \(\eta < \kappa\), let \(U_\eta := \{U : U\ is\ an\ open\ subset\ of\ X\ such\ that\ U \times U \subset W_\eta\}\). Let \(x\) and \(y\) be distinct points of \(X\) and \(\eta < \kappa\) be such that \((x, y)\) is not in \(\overline{W_\eta}\).
Let $U$ and $V$ be open sets in $X$ that contain $x$ and $y$ respectively, and such that $U \times V$ does not intersect $W_\eta$, $U \times U \subset W_\eta$, and $V \times V \subset W_\eta$. To see that no member of $\mathcal{U}_\eta$ intersects both $U$ and $V$, suppose otherwise; that is, suppose that $W$ is a member of $\mathcal{U}_\eta$, $p$ is a point of $W \cap U$ and $q$ is a point of $W \cap V$. Then $(p, q)$ is a point of $W_\eta \cap (U \times V)$, which is a contradiction.

Now, suppose that $\mathcal{U}_\eta$ is a family of open covers of $X$ as described in the lemma. For each $\eta < \kappa$, let $W_\eta = \bigcup_{\eta < \kappa} \{ U \times U : U \in \mathcal{U}_\eta \}$. Clearly, $\Delta_X \subset \bigcap_{\eta < \kappa} W_\eta$. To see that $\Delta_X = \bigcap_{\eta < \kappa} W_\eta$, let $x$ and $y$ be distinct points of $X$. Then there is $\eta < \kappa$ and open sets $U, V \in \mathcal{U}_\eta$ containing $x$ and $y$ respectively, such that no member of $\mathcal{U}_\eta$ intersects both $U$ and $V$. It must be the case that $W_\eta$ does not intersect $U \times V$. □

**Corollary 3.7.** If a space $X$ has a regular $G_\kappa$-diagonal then there is a family $(\mathcal{U}_\eta : \eta < \kappa)$ of open covers of $X$ such that

(a) if $x$ and $y$ are distinct points of $X$, then there exist $\eta < \kappa$ and open sets $U_\eta(x, y), U_\eta(y, x) \in \mathcal{U}_\eta$ containing $x$ and $y$ respectively, such that $U_\eta(y, x) \cap \text{st}(U_\eta(x, y), \mathcal{U}_\eta) = \emptyset$;

(b) if $x \in X$ then $\{ x \} = \bigcap_{\eta < \kappa} \text{st}(x, \mathcal{U}_\eta) = \bigcap_{\eta < \kappa} \overline{\text{st}(x, \mathcal{U}_\eta)}$.

**Proof.** Let $(\mathcal{U}_\eta : \eta < \kappa)$ be a sequence of open covers of $X$ as in Lemma 3.6.

(a) If $x, y \in X$ are distinct points then according to Lemma 3.6 there is $\eta < \kappa$ and open sets $U_\eta(x, y), U_\eta(y, x) \in \mathcal{U}_\eta$ containing $x$ and $y$ respectively, such that no member of $\mathcal{U}_\eta$ intersects both $U_\eta(x, y)$ and $U_\eta(y, x)$. Then $y \in U_\eta(y, x)$ and $U_\eta(y, x) \cap \text{st}(U_\eta(x, y), \mathcal{U}_\eta) = \emptyset$. Therefore $U_\eta(y, x) \cap \overline{\text{st}(U_\eta(x, y), \mathcal{U}_\eta)} = \emptyset$.

(b) Follows immediately from (a). □

It follows from Corollary 3.7 that if a space $X$ has a regular $G_\delta$-diagonal then $\psi_c(X) = \omega$.

**Corollary 3.8 ([3, 6]).** If a space $X$ has a strong rank 2-diagonal, then $X$ has a regular $G_\delta$-diagonal.

**Corollary 3.9 ([3, 6]).** If the rank of the diagonal of a space $X$ is at least 3, then $X$ has a strong rank 2-diagonal.

**Lemma 3.10 ([3]).** Every submetrizable space $X$ has a diagonal of infinite rank.

**Corollary 3.11 ([13]).** Every submetrizable space $X$ has a regular $G_\delta$-diagonal.

**Example 3.12 ([3, Example 2.9]).** There exists a separable Tychonoff Moore space with a regular $G_\delta$-diagonal that is not submetrizable.
Lemma 3.13 ([3]). Every Moore space $X$ has a rank 2-diagonal.

Lemma 3.14 ([13]). Not every Moore space $X$ has a regular $G_δ$-diagonal.

Question 3.15 (A. Bella (see [6], [3])). Is every regular $G_δ$-diagonal a rank 2-diagonal?

As it is noted in [6], there is no example yet even of a space $X$ with a regular $G_δ$-diagonal that does not have a strong rank 2-diagonal.

Conjecture 3.16 ([3]). For every natural number $n$ there is a Tychonoff space $X_n$ with a rank $n$-diagonal that is not a rank $n + 1$-diagonal.

In 1991 Hodel proved the following theorem.

Theorem 3.17 ([16]). If $X$ is a Hausdorff space then $|X| \leq 2^{c(X) H_ψ(X)}$.

A. Bella proved in [7] the following theorem:

Theorem 3.18 ([7]). The cardinality of a ccc-space with a rank 2-diagonal does not exceed $2^ω$.

Therefore if Question 3.15 has a positive answer then Buzyakova’s theorem (Theorem 1.1) will follow from Bella’s theorem (Theorem 3.18).

In [8] Buzyakova asked the following question (in that relation see Example 3.12):

Question 3.19 ([8]). Is there a ccc-space with a regular $G_δ$-diagonal that does not condense onto a first-countable Hausdorff space?

If the answer of the above question is in the negative then Buzyakova’s theorem will follow immediately from Hodel’s inequality (Theorem 3.17).

4. Main results

We begin with a significant generalization of Buzyakova’s result (Theorem 1.1). For that we need the following two simple observations.

Lemma 4.1. Let $X$ be a space such that $wL(X) \leq κ$. Let also $U$ be an open subset of $X$ and $\mathcal{V}$ be an open cover (in $X$) of $\overline{U}$. Then there exists $\mathcal{W} \subset \mathcal{V}$ such that $|\mathcal{W}| \leq κ$ and $U \subset \bigcup \mathcal{W}$.

Proof. Let $X$, $U$ and $\mathcal{V}$ be as hypothesized. Then $\mathcal{V}' := \mathcal{V} \cup \{X \setminus \overline{U}\}$ is an open cover of $X$. Let $\mathcal{W}' \subset \mathcal{V}'$ be such that $|\mathcal{W}'| \leq κ$ and $\overline{\bigcup \mathcal{W}'} = X$. Then $\mathcal{W} := \mathcal{W}' \setminus \{X \setminus \overline{U}\}$ is as required. \qed
Lemma 4.2. Let $X$ be a space with $wL(X) \leq \kappa$, $\mathcal{W}$ be an open cover of $X$ and $\mathcal{U} \subset \mathcal{W}$. Then there exists $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $\overline{\mathcal{U}} \subset \bigcup\{\text{st}(\mathcal{V}, \mathcal{W}) : \mathcal{V} \in \mathcal{V}\}$.

Proof. Let $\mathcal{V}^\prime := \{\text{st}(U, \mathcal{W}) : U \in \mathcal{U}\}$. We claim that $\overline{\mathcal{U}} \subset \bigcup\mathcal{V}^\prime$. Suppose not. Then there is a point $x \in \overline{\mathcal{U}} \setminus \cup \mathcal{V}^\prime$. Let $U_x$ be such that $x \in U_x \in \mathcal{W}$. Then there is $U \in \mathcal{U}$ such that $U_x \cap U \neq \emptyset$. Therefore $U_x \subset \text{st}(U, \mathcal{W})$, hence $x \in \text{st}(U, \mathcal{W})$ – contradiction.

Therefore $\mathcal{V}^\prime$ is a cover of $\overline{\mathcal{U}}$. Then according to Lemma 4.1 there exists $\mathcal{V}_1 \subset \mathcal{V}^\prime$ such that $|\mathcal{V}_1| \leq \kappa$ and $\overline{\mathcal{U}} \subset \bigcup \mathcal{V}_1$. Let $\mathcal{V} := \{V : \text{st}(V, \mathcal{W}) \in \mathcal{V}_1\}$. Then $\mathcal{V}$ is as required. \qed

Theorem 4.3. If $X$ is a Urysohn space then $|X| \leq 2^{wL(X) \cdot \Delta(X)}$.

Proof. Let $\Delta(X) = \kappa$. Then $X$ is a space with a regular $G_\kappa$-diagonal. Therefore, according to Corollary 3.7, there is a family $(\mathcal{U}_\eta : \eta < \kappa)$ of open covers of $X$ such that if $x$ and $y$ are distinct points of $X$, then there exist $\eta < \kappa$ and open sets $U_\eta(x, y), U_\eta(y, x) \in \mathcal{U}_\eta$ containing $x$ and $y$ respectively, such that $U_\eta(y, x) \cap \text{st}(U_\eta(x, y), \mathcal{U}_\eta) = \emptyset$. We denote by $\eta(x, y)$ the minimal such $\eta$ and let $Y_\eta(x) := \{y : y \in X \setminus \{x\}, \eta(x, y) = \eta\}$.

Let $wL(X) = \tau$. Then for each $\eta < \kappa$ we can find a subfamily $\mathcal{D}_\eta$ of $\mathcal{U}_\eta$ such that $|\mathcal{D}_\eta| \leq \tau$ and $X = \overline{\bigcup \mathcal{D}_\eta}$. Also, for each $x \in X$ and each $\eta < \kappa$ there is $Y_\eta^\prime(x) \subset Y_\eta(x)$ such that $|Y_\eta^\prime(x)| \leq \tau$ and $\mathcal{W}_{x, \eta} := \{\text{st}(U_\eta(y, x), \mathcal{U}_\eta) : y \in Y_\eta^\prime(x)\}$ is such that $\bigcup\{U_\eta(y, x) : y \in Y_\eta(x)\} \subset \bigcup \mathcal{W}_{x, \eta}$ (Lemma 4.2).

For each $x \in X$ and each $\eta < \kappa$, let $\mathcal{G}_x^\eta \subset \mathcal{D}_\eta$ be such that $G \in \mathcal{G}_x^\eta$ if and only if $\bigcup \mathcal{W}_{x, \eta} \cap G \neq \emptyset$. Then $|\mathcal{G}_x^\eta| \leq \tau$. Also, for each $y \in Y_\eta^\prime(x)$ we define $\mathcal{G}_x^\eta_{y, \xi} \subset \mathcal{D}_\eta$ to be such that $G \in \mathcal{G}_x^\eta_{y, \xi}$ if and only if $G \cap \text{st}(U_\eta(y, x), \mathcal{U}_\eta) \neq \emptyset$. Clearly $\mathcal{G}_x^\eta_{y, \xi} \subset \mathcal{G}_x^\eta$, hence $|\mathcal{G}_x^\eta_{y, \xi}| \leq \tau$ and since each $G \in \mathcal{D}_\eta$ is open we have $\mathcal{G}_x^\eta = \bigcup\{\mathcal{G}_x^\eta_{y, \xi} : y \in Y_\eta^\prime(x)\}$.

Now for each $y \in Y_\eta(x)$ let $\mathcal{G}_x^{\eta, y} \subset \mathcal{D}_\eta$ be such that $G \in \mathcal{G}_x^{\eta, y}$ if and only if $U_\eta(y, x) \cap G \neq \emptyset$ and for each $y \in Y_\eta^\prime(x)$ let $\mathcal{G}_x^{\eta, y}_{\xi, \eta} \subset \mathcal{D}_\eta$ contains all $G \in \mathcal{G}_x^{\eta, y}$ such that $G \in \mathcal{G}_x^{\eta, y}$ and $U_\eta(y, x) \subset \bigcup \mathcal{W}_{x, \eta}$. Since $U_\eta(y, x) \subset \bigcup \mathcal{W}_{x, \eta}$, we have $\mathcal{G}_x^{\eta, y} = \bigcup\{\mathcal{G}_x^{\eta, y}_{\xi, \eta} : y \in Y_\eta^\prime(x)\}$. Now from $X = \bigcup \mathcal{D}_\eta$ it follows that for each $y \in Y_\eta(x)$ we have $y \in \bigcup \mathcal{G}_x^{\eta, y}$ and clearly $x \in X \setminus \bigcup \mathcal{G}_x^{\eta, y}$. Let $N_\eta(x) := \bigcap\{X \setminus \bigcup \mathcal{G}_x^{\eta, y} : y \in Y_\eta(x)\}$. Then $x \in N_\eta(x)$ and $N_\eta(x) \cap Y_\eta(x) = \emptyset$. Therefore $\{x\} = \bigcap\{N_\eta(x) : \eta < \kappa\}$.

For each $\eta < \kappa$ there are at most $2^\tau$ possible subsets $\mathcal{G}_x^\eta$ of $\mathcal{D}_\eta$ with cardinality $\leq \tau$ for $|\mathcal{D}_\eta| \leq \tau$. Each such subset $\mathcal{G}_x^\eta$ could be represented as $\mathcal{G}_x^\eta = \bigcup\{\mathcal{G}_x^{\eta, y}_{\xi, \eta} : y \in Y_\eta^\prime(x)\}$ in $(2^\tau)^{\tau} = 2^\tau$ different ways. For each such representation there are at most $(2^\tau)^{\tau} = 2^\tau$ possible unions of the type $\mathcal{G}_x^{\eta, y} = \bigcup\{\mathcal{G}_x^{\eta, y}_{\xi, \eta} : y \in Y_\eta^\prime(x)\}$. Thus, for each $\eta < \kappa$ there are at most...
2^\tau \cdot 2^\tau \cdot 2^\tau = 2^\tau \text{ possible different intersections } N_\eta(x). \text{ Hence, there are at most } (2^\tau)^\kappa \text{ possible different intersections } \cap \{N_\eta(x) : \eta < \kappa \}. \text{ Since each such intersection is equal to exactly one point } x \in X \text{ we conclude that } |X| \leq 2^{\tau \cdot \kappa}.

\textbf{Corollary 4.4.} If } X \text{ is a Urysohn space then } |X| \leq 2^{c(X) \cdot \Xi(X)}.

Note that Corollary 4.4 directly generalizes Theorem 1.1 for spaces with regular } G_\kappa \text{-diagonals.}

\textbf{Corollary 4.5.} If } X \text{ is a Urysohn space then } |X| \leq 2^{aL(X) \cdot \Xi(X)}.

\textbf{Corollary 4.6.} If } X \text{ is a Urysohn space then } |X| \leq 2^{\chi(X) \cdot wL(X) \cdot \Xi(X)}.

\textbf{Corollary 4.7.} If } X \text{ is a Urysohn space and } \chi(X) \cdot wL(X) \geq \Xi(X) \text{ then } |X| \leq 2^{\chi(X) \cdot wL(X)}.

Note that Corollary 4.7 shows in particular that Bell, Ginsburg and Woods inequality is true for all spaces with regular } G_\delta \text{-diagonals.

\textbf{Corollary 4.8.} If } X \text{ is a space with a regular } G_\delta \text{-diagonal then } |X| \leq 2^{wL(X)}.

\textbf{Corollary 4.9.} If } X \text{ is a weakly Lindelöf space with a regular } G_\delta \text{-diagonal then } |X| \leq 2^\omega.

In particular Corollary 4.9 applies for all submetrizable spaces and spaces with strong rank 2-diagonals.

\textbf{Corollary 4.10.} If } X \text{ is an almost Lindelöf space with a regular } G_\delta \text{-diagonal then } |X| \leq 2^\omega.

In relation to Corollary 4.9, Theorem 3.18, Lemma 3.14 and Lemma 3.13 it is natural to ask the following question:

\textbf{Question 4.11.} Is it true that the cardinality of every weakly Lindelöf Moore space does not exceed } 2^\omega ?

And a more general question:

\textbf{Question 4.12.} Is it true that the cardinality of every weakly Lindelöf space with a rank 2-diagonal does not exceed } 2^\omega ?

With the next theorem we significantly improve the result in Corollary 4.6.

\textbf{Theorem 4.13.} Let } X \text{ be a Urysohn space. Then } |X| \leq wL(X)^{\chi(X) \cdot \Xi(X)}.
Proof. Let $\Delta(X) = \kappa$. Then $X$ is a space with a regular $G_\kappa$-diagonal. Therefore, according to Lemma 3.6, there is a family $(U_\eta : \eta < \kappa)$ of open covers of $X$ such that if $x$ and $y$ are distinct points of $X$, then there exists $\eta < \kappa$ and open sets $U_\eta(x, y), U_\eta(y, x) \in U_\eta$ containing $x$ and $y$ respectively, such that no member of $U_\eta$ intersects both $U_\eta(x, y)$ and $U_\eta(y, x)$.

Let $wL(X) \leq \mu$. Then for each $\eta < \kappa$ we can find a subfamily $D_\eta$ of $U_\eta$ such that $|D_\eta| \leq \mu$ and $X = \bigcup D_\eta$. Let $D := \bigcup \{D_\eta : \eta < \kappa\}$. Then clearly $|D| \leq \kappa \cdot \mu$.

Let $\chi(X) \leq \tau$. Then for every $x \in X$ there exists a family $V_x := \{V_\xi(x) : \xi < \tau\}$ of open sets in $X$ which is a basis at $x$.

For every $x \in X$, $\eta < \kappa$ and $\xi < \tau$ we choose $D_{\eta, \xi}(x) \in D_\eta$ such that $D_{\eta, \xi}(x) \cap V_\xi(x) \neq \emptyset$. Such $D_{\eta, \xi}(x)$ exists for $X = \bigcup D_\eta$. Therefore the map

$$F(x)(\eta, \xi) = \{D_{\eta, \xi}(x) : D_{\eta, \xi}(x) \in D_\eta, D_{\eta, \xi}(x) \cap V_\xi(x) \neq \emptyset\}$$

from $X$ to $[D]^{\tau \cdot \kappa}$ is well-defined. To show that $F$ is an injection, let $x, y \in X$ be two distinct points. Then there exists $\eta < \kappa$ and open sets $U_\eta(x, y), U_\eta(y, x) \in U_\eta$ containing $x$ and $y$ respectively, such that no member of $U_\eta$ intersects both $U_\eta(x, y)$ and $U_\eta(y, x)$. Since $V_x$ is a basis at $x$ there exists $\xi < \tau$ such that $V_\xi(x) \subset U_\eta(x, y)$. Let $D = F(x)(\eta, \xi)$. Then $D \cap V_\xi(x) \neq \emptyset$ and therefore $D \cap U_\eta(x, y) \neq \emptyset$. Since $D \in D_\eta \subset U_\eta$, we have $D \cap U_\eta(y, x) = \emptyset$. Therefore $F(y)(\eta, \xi) \neq D$. □

**Corollary 4.14.** Let $X$ be a space with a regular $G_\delta$-diagonal. Then $|X| \leq wL(X)^{\chi(X)}$.

**Corollary 4.15.** Let $X$ be a first countable space with a regular $G_\delta$-diagonal. Then $|X| \leq wL(X)^{\omega}$.

**Corollary 4.16.** If $X$ is a weakly Lindelöf, first countable space then $|X| \leq 2^{\Delta(X)}$.

**Remarks 4.17.**

1. **Corollary 4.15 should be compared with the following Proposition 4.7 from [6]: If $X$ has a rank 3-diagonal then $|X| \leq wL(X)^{\omega}$.**

2. **In [6] the authors asked the following question: Is it the case that if $X$ has a strong rank 2-diagonal then $|X| \leq wL(X)^{\omega}$?**

We finish with a theorem that significantly improves the result in Corollary 4.5.

**Theorem 4.18.** If $X$ is a Urysohn space then $|X| \leq aL(X)^{\Delta(X)}$. 
Proof. Let $\Delta(X) = \kappa$. Then $X$ is a space with a regular $G_\kappa$-diagonal. Therefore, according to Corollary 3.7, there is a family $(U_\eta : \eta < \kappa)$ of open covers of $X$ such that if $x$ and $y$ are distinct points of $X$, then there is $\eta < \kappa$ and an open set $U_\eta(x,y) \in U_\eta$ containing $x$ such that $y \notin \text{st}(U_\eta(x,y), U_\eta)$.

Let $aL(X) = \tau$. Then for each $\eta < \kappa$ we can find a subfamily $D_\eta$ of $U_\eta$ such that $|D_\eta| \leq \tau$ and $X = \cup \{\overline{D} : D \in D_\eta\}$.

Let $x \in X$. For each $\eta < \kappa$ we fix $D_{x,\eta} \in D_\eta$ such that $x \in \overline{D}_{x,\eta}$. Now, let $y \in X \setminus \{x\}$. Then there is $\eta < \kappa$ and an open set $U_\eta(x,y) \in U_\eta$ containing $x$ such that $y \notin \text{st}(U_\eta(x,y), U_\eta)$. Since $D_{x,\eta} \in D_\eta \subset U_\eta$, we have $D_{x,\eta} \subset \text{st}(U_\eta(x,y), U_\eta)$. Hence $\overline{D}_{x,\eta} \subset \text{st}(U_\eta(x,y), U_\eta)$ and therefore $y \notin \overline{D}_{x,\eta}$. This shows that $\{x\} = \cap_{\eta<\kappa} \overline{D}_{x,\eta}$.

Since each $D_{x,\eta}$ could be chosen out of $\tau$ many sets, there are $\tau^\kappa$ such possible intersections. Therefore we conclude that $|X| \leq \tau^\kappa$. \qed

Corollary 4.19. If $X$ is a space with a regular $G_\delta$-diagonal then $|X| \leq aL(X)^\omega$.

Note that Corollary 4.19 also follows from Corollary 4.15 and that as a corollary of the previous theorem we again obtain Corollary 4.10. Also, the result in Theorem 4.18 and in Corollary 4.15 strengthen, for the class of spaces with regular $G_\delta$-diagonals, the following Bella and Cammaroto result: Let $X$ be a Urysohn space. Then $|X| \leq 2^{\chi(X) \cdot aL(X)}$ [5].

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References


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