GENERALIZATIONS OF TWO CARDINAL INEQUALITIES OF HAJNAL AND JUHÁSZ

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Abstract. A non-empty subset $A$ of a topological space $X$ is called finitely non-Hausdorff if for every non-empty finite subset $F$ of $A$ and every family $\{U_x : x \in F\}$ of open neighborhoods $U_x$ of $x \in F$, $\cap\{U_x : x \in F\} \neq \emptyset$ and the non-Hausdorff number $nh(X)$ of $X$ is defined as follows: $nh(X) := 1 + \sup\{|A| : A \subset X$ is finitely non-Hausdorff$\}$. Clearly, if $X$ is a Hausdorff space then $nh(X) = 2$.

We define the non-Urysohn number of $X$ with respect to the singletons, $nu_s(X)$, as follows: $nu_s(X) := 1 + \sup\{\cl_\theta(\{x\}) : x \in X\}$.

In 1967 Hajnal and Juhász proved that if $X$ is a Hausdorff space then: (1) $|X| \leq 2^{c(X)} \chi(X)$; and (2) $|X| \leq 2^{2^{s(X)}}$; where $c(X)$ is the cellularity, $\chi(X)$ is the character and $s(X)$ is the spread of $X$.

In this paper we generalize (1) by showing that if $X$ is a topological space then $|X| \leq nh(X)c(X)\chi(X)$. Immediate corollary of this result is that (1) holds true for every space $X$ for which $nh(X) \leq 2^\omega$ (and even for spaces with $nh(X) \leq 2^{c(X)}\chi(X)$). This gives an affirmative answer to a question posed by M. Bonanzinga in 2013. A simple example of a $T_1$, first countable, ccc-space $X$ is given such that $|X| > 2^\omega$ and $|X| = nh(X)^\omega = nh(X)$. This example shows that the upper bound in our inequality is exact and that $nh(X)$ cannot be omitted (in particular, $nh(X)$ cannot always be replaced by 2 even for $T_1$-spaces).

In this paper we also generalize (2) by showing that if $X$ is a $T_1$-space then $|X| \leq 2^{nu_s(X)}2^{2^{s(X)}}$. It follows from our result that (2) is true for every $T_1$-space for which $nu_s(X) \leq 2^{s(X)}$. A simple example shows that the presence of the cardinal function $nu_s(X)$ in our inequality is essential.

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1. Introduction

Throughout this paper \( \omega \) is (the cardinality of) the set of non-negative integers, \( \xi, \eta \) and \( \alpha \) are ordinals and \( \kappa \) and \( \tau \) are infinite cardinals. The cardinality of the set \( X \) is denoted by \(|X|\). By space we mean infinite topological space and for a subset \( U \) of a space \( X \) the closure of \( U \) is denoted by \( \overline{U} \).

Recall that a pairwise disjoint collection of non-empty open sets in a space \( X \) is called a cellular family. The cellularity of \( X \) is \( c(X) := \sup\{|U| : U \text{ a cellular family in } X\} + \omega \). If \( c(X) = \omega \) then it is called that \( X \) satisfies the countable chain condition (or ccc) property. For \( x \in X \) the character of \( X \) at the point \( x \) is \( \chi(x, X) := \min\{|B| : B \text{ is a local base for } x\} \) and the character of \( X \) is \( \chi(X) := \sup\{\chi(x, X) : x \in X\} \).

In what follows, whenever \( X \) is a space with \( \chi(X) = \tau \) we shall assume that for each \( x \in X \) a local base \( B_x \) with \( |B_x| \leq \kappa \) has been fixed and if \( A \subseteq X \) then by \( U_A \) we shall denote the set of all families \( U := \{B_x : x \in A, B_x \in B_x\} \).

The following two definitions appeared in [2].

**Definition 1.1.** A non-empty subset \( A \) of a topological space \( X \) is called finitely non-Hausdorff if, for every non-empty finite subset \( F \) of \( A \) and every \( U \in U_F \), \( \cap U \neq \emptyset \). The set \( A \) is called a maximal finitely non-Hausdorff subset of \( X \) if \( A \) is a finitely non-Hausdorff subset of \( X \) and if \( B \) is a finitely non-Hausdorff subset of \( X \) such that \( A \subseteq B \) then \( A = B \).

**Definition 1.2.** Let \( X \) be a topological space. The non-Hausdorff number \( nh(X) \) of \( X \) is defined as follows: \( nh(X) := 1 + \sup\{|A| : A \text{ is a (maximal) finitely non-Hausdorff subset of } X\} \).

M. Bonanzinga introduced in [1] (independently from [2]) the notion of a Hausdorff number of a topological space \( X \), denoted \( H(X) \), as follows: \( H(X) := \min\{\tau : \text{for every } A \subseteq X \text{ with } |A| \geq \tau \text{ there exist } U \in U_A \text{ such that } \cap U = \emptyset\} \) and she called \( n \)-Hausdorff every space \( X \) with \( H(X) \leq n \), where \( n \in \omega \) and \( n \geq 2 \). It follows immediately from the definitions of \( H(X) \) and \( nh(X) \) that if \( n \in \omega \) and \( n \geq 2 \) then \( H(X) = n \) if and only if \( nh(X) = n \). In the same paper Bonanzinga also introduced the notion of a weak Hausdorff number, denoted \( H^*(X) \), as follows: \( H^*(X) := \min\{\tau : \text{for every } A \subseteq X \text{ such that } |A| \geq \tau \text{ there exist } B \subseteq A \text{ with } |B| < \tau \text{ and } U \in U_B \text{ such that } \cap U = \emptyset\} \). She noted there that for every space \( X \), \( H^*(X) = H(X) \) or \( H^*(X) = H(X)^+ \) and constructed an example of a space \( X \) such that \( H^*(X) = H(X) = \omega \) (hence \( H(X) \neq n \) for every \( n < \omega \)). It follows from the definitions that if \( H^*(X) \leq \omega \) then either \( nh(X) = H(X) = n \) for some \( n < \omega \).
or \( H(X) = \omega \) and \( X \) is such that for every \( n \in \omega \), \( n \geq 2 \) there is a finitely non-Hausdorff subset \( A \) of \( X \) with \( |A| = n \) but there does not exist countably infinite finitely non-Hausdorff subset of \( X \). Therefore if \( H^*(X) \leq \omega \) then \( nh(X) \leq \omega \). (Clearly, it is possible \( nh(X) = \omega \) and \( H^*(X) > \omega \)).

In 1967, Hajnal and Juhász proved that if \( X \) is a Hausdorff space then \( |X| \leq 2^{c(X)\chi(X)} \) (see [4], [6] or [5]). Recently M. Bonanzinga showed that \( |X| \leq 2^{c(X)\chi(X)} \) whenever \( X \) is a 3-Hausdorff space ([1, Corollary 54]) and she asked if the much more stronger inequality \( |X| \leq 2^{c(X)\chi(X)} \) holds true for every space \( X \) with a finite Hausdorff number ([1, Question 55]).

In this paper we prove that if \( X \) is a topological space then \( |X| \leq nh(X)c(X)\chi(X) \). Immediate corollary of this result is that the Hajnal–Juhász inequality \( |X| \leq 2^{c(X)\chi(X)} \) holds true for every space \( X \) for which \( nh(X) \leq 2^\omega \) (and even for spaces for which \( nh(X) \leq 2^{c(X)\chi(X)} \)). This gives an affirmative answer of Bonanzinga’s question. An Example of a \( T_1 \), first countable, ccc-space \( X \) is given such that \( |X| > 2^\omega \) and \( |X| = nh(X)^\omega = nh(X) \). This example shows that the upper bound in our inequality is exact and that \( nh(X) \) cannot be omitted (in particular, \( nh(X) \) cannot always be replaced by 2 even for \( T_1 \)-spaces).

2. Some observations about \( nh(X) \) and finitely non-Hausdorff subsets of \( X \)

As it was noted in [2], it follows immediately from Definition 1.2 that \( X \) is a Hausdorff space if and only if \( nh(X) = 2 \) and \( 2 < nh(X) \leq 1 + |X| \) whenever \( X \) is a non-Hausdorff space. Also, if \( X \) is a topological space and \( A \subset X \), then \( nh(A) \leq nh(X) \), and if \( X \) is an infinite set with topology generated by the open sets \( \{X \setminus \{x\}: x \in X\} \), then \( X \) is a maximal finitely non-Hausdorff set, and therefore \( nh(X) = |X| \).

The following three observations follow immediately from the definitions.

**Proposition 2.1** ([2]). In a Hausdorff space \( X \) the only maximal finitely non-Hausdorff subsets of \( X \) are the singletons.

**Proposition 2.2** ([2]). Every finitely non-Hausdorff subset of a topological space \( X \) is contained in a maximal finitely non-Hausdorff subset of \( X \).

**Proposition 2.3.** Every subset of a finitely non-Hausdorff subset of a space \( X \) is a finitely non-Hausdorff subset of \( X \).

Having in mind Propositions 2.1 and 2.3 one can easily construct an example of a \( T_1 \)-space \( X \) and subsets \( A \) and \( B \) of \( X \) such that \( A \subset B \),
A and B are finitely non-Hausdorff subsets of X but A is not finitely non-Hausdorff subset of B (e.g. take $B = \alpha$ in Example 3.3 and let A be any subset of B which is not a singleton).

The following two observations appeared in [2]. Since we are going to use them later, we give them here with their proofs.

**Lemma 2.4 ([2]).** Let $X$ be a space and $A$ be a finitely non-Hausdorff subset of $X$. Then $A \subset \cap\{\cap U : U \in U_F, \emptyset \neq F \subset A, |F| < \omega\}$.

**Proof.** Let $F$ be a non-empty subset of $A$, $U_0 \in U_F$, and $G = \cap U_0$. Suppose that there exists $a_0 \in A$ such that $a_0 \notin \overline{G}$. Then there is $W_{a_0} \in N_{a_0}$ such that $W_{a_0} \cap G = \emptyset$. Let $V_{a_0} = W_{a_0}$ if $a_0 \notin F$ and $V_{a_0} = U_{a_0} \cap W_{a_0}$, where $U_{a_0} \in \mathcal{U}_0$ and $U_{a_0} \in N_{a_0}$ if $a_0 \in F$. Then the family $U_1 := \{V_{a_0}\} \cup \{U_a : U_a \in U_0, a \in F \setminus \{a_0\}\}$ has the property that $\cap U_1 = \emptyset$, a contradiction. Therefore $A \subset \overline{U}$ for every $U \in U_F$ and every non-empty subset $F$ of $A$ with $|F| < \omega$. Thus, $A \subset \cap\{\cap U : U \in U_F, \emptyset \neq F \subset A, |F| < \omega\}$. □

**Theorem 2.5 ([2]).** Let $X$ be a space and $A$ be a maximal finitely non-Hausdorff subset of $X$. Then $A = \cap\{\cap U : U \in U_F, \emptyset \neq F \subset A, |F| < \omega\}$.

**Proof.** Let $A$ be a maximal finitely non-Hausdorff subset of $X$. Then it follows from Lemma 2.4 that $A \subset \cap\{\cap U : U \in U_F, \emptyset \neq F \subset A, |F| < \omega\}$. Suppose that there is $x_0 \in \cap\{\cap U : U \in U_F, \emptyset \neq F \subset A, |F| < \omega\} \setminus A$. Then $U \cap (\cap U \neq \emptyset$ for every $U \in N_{x_0}$, every $U \in U_F$, and every non-empty finite subset $F$ of $A$. Thus, for the set $A_1 := A \cup \{x_0\}$, we have that if $F \subset A_1$ with $F \neq \emptyset$ and $|F| < \omega$ and $U \in U_F$, then $\cap U \neq \emptyset$. Therefore, $A_1$ is a finitely non-Hausdorff subset of $X$ and $A \subseteq A_1$, a contradiction with the maximality of $A$. □

**Corollary 2.6.** Every maximal finitely non-Hausdorff subset of a space $X$ is a closed set.

**Corollary 2.7.** Let $X$ be a space and $A$ be a finitely non-Hausdorff subset of $X$. Then $\overline{A}$ is a finitely non-Hausdorff subset of $X$.

**Proof.** It follows immediately from Proposition 2.2, Corollary 2.6 and Proposition 2.3. □

**Corollary 2.8.** Let $X$ be a space and $A$ be a finitely non-Hausdorff subset of $X$. If $x \in A$ then $A \subset \cap\{\overline{B} : B \in \mathcal{B}_x\}$, hence $A \subset \cap_{x \in A}(\cap\{\overline{B} : B \in \mathcal{B}_x\})$.

In relation to Corollary 2.8 we can say more.
Lemma 2.9. Let $X$ be a space and $x \in X$. Then $\bigcap \{B : B \in B_x\} = \bigcup \{M : M \text{ is a (maximal) finitely non-Hausdorff subset of } X \text{ that contains } x\}$.

Proof. Let $y \in \cap \{B : B \in B_x\}$ and $U$ be an open neighborhood of $y$. Then $U \cap B \neq \emptyset$ for every $B \in B_x$. Therefore the set $\{x, y\}$ is a finitely non-Hausdorff subset of $X$, hence it is contained in some maximal one. Therefore $\cap \{B : B \in B_x\} \subseteq \bigcup \{M : M \text{ is a (maximal) finitely non-Hausdorff subset of } X \text{ that contains } x\}$.

Now let $y \in \bigcup \{M : M \text{ is a (maximal) finitely non-Hausdorff subset of } X \text{ that contains } x\}$. Then there exists a (maximal) finitely non-Hausdorff subset $M_y$ of $X$ such that $y \in M_y$. Then the set $\{x, y\} \subseteq M_y$ is a finitely non-Hausdorff subset of $X$ (Proposition 2.3). Thus if $B \in B_x$ and $U$ is an arbitrary open neighborhood of $y$ we have $B \cap U \neq \emptyset$. Hence $y \in B$ and therefore $y \in \cap \{B : B \in B_x\}$. $\square$

Corollary 2.10. Let $X$ be a space and $A$ be a finitely non-Hausdorff subset of $X$. Then $\cap_{x \in A}(\cap \{B : B \in B_x\}) = \cap_{x \in A}(\bigcup \{M : M \text{ is a (maximal) finitely non-Hausdorff subset of } X \text{ that contains } x\})$.

The following example shows that the intersection in Corollary 2.10 could be different from $A$ even when $A$ is a maximal non-Hausdorff subset of $X$.

Example 2.11. There exists a first countable $T_1$-space $X$ and a maximal finitely non-Hausdorff subset $A$ of $X$ such that

$$A \subseteq \cap_{x \in A}(\cap \{B : B \in B_x\}).$$

Proof. Let $\{b\}$, $A := \{a_i : i \in \omega\}$, $S := \{n : n \in \omega\}$, and $N^2 := \{(i, n) : i, n \in \omega\}$ be pairwise disjoint sets and $X := A \cup \{b\} \cup S \cup N^2$. We define topology on $X$ as follows: all points in $S \cup N^2$ are isolated (hence each one is an open and closed set); the points in $S$ form a convergent sequence that converges to the point $a_i$ for every $i \in \omega$; for each $i \in \omega$ the points $\{(i, n) : n \in \omega\}$ form a convergent sequence that approaches to the points $a_i$ and $b$. In order $X$ to be first countable we also require the set $\{b\} \cup \{(i, n) : i \in \omega : n \geq k\} : k \in \omega$ to form an open basis for the topology at $b$. Then $A$ and each of the sets $\{a_i, b\}$, $i \in \omega$, are maximal finitely non-Hausdorff subsets of $X$, $A \cup \{b\}$ is not a finitely non-Hausdorff subset of $X$ and $A \not\subseteq A \cup \{b\}$.

We recall that the $\theta$-closure of a set $A$ in a space $X$ is the set $\text{cl}_\theta(A) : = \{x \in X : \text{ for every } B \in B_x, \overline{B} \cap A \neq \emptyset\}$.

Proposition 2.12. Let $X$ be a space and $x \in X$. Then $\text{cl}_\theta(\{x\}) = \cap \{\overline{B} : B \in B_x\}$.
Corollary 2.13. Let $X$ be a space and $x \in X$. Then $\text{cl}_0(\{x\}) = \bigcup \{M : M \text{ is a (maximal) finitely non-Hausdorff subset of } X \text{ that contains } x\}$.

For convenience we introduce the following notation to be used in the proof of our main result.

Notation 2.14. Let $X$ be a space and $A \subseteq X$. Then

$$\mathcal{F}_A := \{F : F \subset A, F \text{ is a finite, finitely non-Hausdorff subset of } X\}.$$ 

Using this notation we can restate Corollary 2.13 as follows:

Corollary 2.15. Let $X$ be a space and $x \in X$. Then $\text{cl}_0(\{x\}) = \bigcup \{F : F \in \mathcal{F}_X, x \in F\}$.

Corollary 2.16. Let $X$ be a space and $x \in X$. The union of all (maximal) finitely non-Hausdorff subsets of $X$ that contain $x$ is a closed set in $X$.

We recall that a non-empty subset $A$ of a topological space $X$ is called finitely non-Urysohn (see [3]) if for every non-empty finite subset $F$ of $A$ and every family $\{U_x : x \in F\}$ of open neighborhoods $U_x$ of $x \in F$, $\cap\{U_x : x \in F\} \neq \emptyset$ and the non-Urysohn number of $X$ is defined as follows: $\nu(X) := 1 + \sup\{|A| : A \text{ is a finitely non-Urysohn subset of } X\}$.

Corollary 2.17. Let $X$ be a space and $x \in X$. Then $\text{cl}_0(\{x\})$ is a finitely non-Urysohn subset of $X$ that contains $x$.

Corollary 2.18. Let $X$ be a space and $x \in X$. Then $\nu(X) \geq |\bigcup \{M : M \text{ is a maximal finitely non-Hausdorff subset of } X \text{ that contains } x\}|$.

Corollary 2.19. If $X$ is a space then $\nu(X) \geq \text{nh}(X)$.

We finish this section with one more observation.

Lemma 2.20. Let $X$ be a space and $x \in X$ be a point such that $\overline{U} = X$ whenever $U \subset X$ is an open neighborhood of $x$. If $M$ is a maximal finitely non-Hausdorff subset of $X$ then $x \in M$.

Proof. Let $M$ be a maximal finitely non-Hausdorff subset of $X$. Suppose that $x \notin M$. Then there exist a finite set $F \subset M$, $U \in \mathcal{U}_F$ and an open neighborhood $U$ of $x$ such that $\cap U \neq \emptyset$. Since $M$ is a finitely non-Hausdorff subset of $X$, $\cap U \neq \emptyset$. Let $y \in \cap U$. Then $y \notin \overline{U}$ - contradiction. \qed
3. More cardinal inequalities involving the non-Hausdorff number

The following theorem generalizes Hajnal–Juhász inequality that if $X$ is a Hausdorff space then $|X| \leq 2^{c(X)\chi(X)}$.

**Theorem 3.1.** Let $X$ be a space. Then $|X| \leq nh(X)^{c(X)\chi(X)}$.

**Proof.** Let $c(X)\chi(X) = \kappa$, $nh(X) = \tau$ and for each $x \in X$ let $B_x$ be a local base for $x$ in $X$ with $|B_x| \leq \kappa$. Let also $x_0$ be an arbitrary point in $X$. We construct a sequence $\{G_\eta : \eta < \kappa^+\}$ of subsets of $X$ such that

1. $G_0 = \{x_0\}$;
2. $\cup_{\xi < \eta} G_\xi \subset G_\eta$ and $|G_\eta| \leq \tau^\kappa$ for every $0 < \eta < \kappa^+$;
3. If $\eta$ is a limit ordinal then $G_\eta = \cup_{\xi < \eta} G_\xi$;
4. If $x \in G_\eta$ then there exists a maximal finitely non-Hausdorff subset $M_x$ of $X$ such that $M_x \subset G_{\eta+1}$;
5. If $\{W_\xi : \xi < \kappa\}$ is a collection of $\leq \kappa$ open sets in $X$ such that $W_\xi = \cup_{\alpha < \kappa} G_\alpha$, where each $G_\alpha = \cap_{U \in U_\alpha} F_\xi, \alpha$, for some $U_\alpha, \alpha \in U_{\xi,\alpha}$ with $F_\xi,\alpha \in F_{G_\alpha}$, and $\cup_{\xi < \kappa} W_\xi \neq X$, then $G_{\eta+1} \setminus (\cup_{\xi < \kappa} W_\xi) \neq \emptyset$.

Let $G = \cup_{\xi < \kappa^+} G_\eta$. If $G = X$ then the proof is complete. Suppose there is $y \in X \setminus G$. Let $B_y = \{B_\xi : \xi < \kappa\}$ be a local base at $y$. For each $\xi < \kappa$ let $W_\xi = \{\cap U : U \in U_F, F \in F_G, (\cap U) \cap B_\xi = \emptyset\}$. Then clearly $y \notin \cup_{\xi < \kappa} \overline{W_\xi}$.

**Claim 1:** For each $x \in G$, there exist $F \in F_G$ with $x \in F$, $\cup \in U_F$ and $\xi < \kappa$ such that $(\cap U) \cap B_\xi = \emptyset$.

**Proof:** Let $x \in G$. Then there is $\eta < \kappa^+$ such that $x \in G_\eta$. It follows from (4) that there exists a maximal finitely non-Hausdorff subset $M_x$ of $X$ such that $x \in M_x \subset G_{\eta+1}$. Since $y \notin G$ we have $y \notin M_x$. Thus, there exists a finite set $F_x \subset M_x$, $U' \in U_{F_x}$ and $\xi < \kappa$ such that $(\cap U') \cap B_\xi = \emptyset$. Therefore for the finitely non-Hausdorff subset $F := F_x \cup \{x\} \subset M_x$ of $F_G$ there exists $U \in U_F$ such that $(\cap U) \cap B_\xi = \emptyset$. \hfill \Box

**Claim 2:** $G \subseteq \cup_{\xi < \kappa} \overline{W_\xi}$.

**Proof:** Let $x \in G$. It follows from Claim 1 that there exists $F \in F_G$ with $x \in F$, $U \in U_F$ and $\xi < \kappa$ such that $(\cap U) \cap B_\xi = \emptyset$. Then it follows from Lemma 2.4 that $F \subset \cap U$, hence $x \in \overline{W_\xi}$ and therefore $x \in \cup_{\xi < \kappa} \overline{W_\xi}$.

Since $c(X) \leq \kappa$, there exists $G_\xi \subseteq W_\xi$ with $|G_\xi| \leq \kappa$ such that $\cup W_\xi \subseteq \overline{G_\xi}$. Let $W_\xi = \cup G_\xi$. Then $G \subseteq \cup_{\xi < \kappa} \overline{W_\xi}$ and $y \notin \cup_{\xi < \kappa} \overline{W_\xi}$.
Let $\mathcal{F} := \{ F : \text{there is } U \in \mathcal{U}_F \text{ and } \xi < \kappa \text{ such that } \bigcap U \in \mathcal{G}_\xi \}$. Since $|\mathcal{F}| \leq \kappa$, we have $| \bigcup \mathcal{F} | \leq \kappa$. Clearly $\bigcup \mathcal{F} \subset \mathcal{G}$. Thus, we can find $\eta < \kappa$ such that $\bigcup \mathcal{F} \subset \mathcal{G}[\eta]$. Then it follows from (5) that $G[\eta+1] \setminus (\bigcup_{\xi<\kappa} W[\xi]) \neq \emptyset$. This contradicts $G \subseteq \bigcup_{\xi<\kappa} W[\xi]$. \hfill $\square$

**Corollary 3.2.** Let $X$ be a space with $nh(X) \leq 2^{c(X)}\chi(X)$. Then $|X| \leq 2^{c(X)\chi(X)}$.

Corollary 3.2 answers in the affirmative the following question of Bonanzainga (see [1, Question 55]): Is $|X| \leq 2^{c(X)\chi(X)}$ true for every $X$ such that $H(X)$ is finite? It also greatly improves her Corollary 54 that states that for every 3-Hausdorff space $X$, $|X| \leq 2^{c(X)\chi(X)}$.

The following example shows that Hajnal–Juhász inequality is not always true for $T_1$-spaces and that the cardinal number $nh(X)$ in the inequality in Theorem 3.1 cannot be replaced by 2. For a different example of a $T_1$-space for which Hajnal–Juhász inequality is not true see [1, Example 13].

**Example 3.3** (see [2, Example 2.1]). Let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{R}$ be the set of all real numbers. Let $S := \{1/n : n \in \mathbb{N}\}$ and $M := S \cup \{0\}$ be the subspace of $\mathbb{R}$ with the inherited topology. Then in $M$ all points except 0 are isolated and $\lim_{n \to \infty} 1/n = 0$. Let $\alpha$ be an infinite initial ordinal. We duplicate $\alpha$ many times the point 0 $\in M$; i.e. we replace in $M$ the point 0 with $\alpha$ many distinct points and obtain the set $X := S \cup \alpha$ with topology such that, for each $\beta < \alpha$, we have $\beta \in \lim_{n \to \infty} 1/n$ and the subspaces $S$ and $\alpha$ with the inherited topology from $X$ are discrete. Then $X$ is $T_1$ (but not Hausdorff) ccc-space, $\chi(X) = \omega$, and $nh(X) = \alpha$. Therefore if $\alpha > 2^\omega$ is a cardinal for which $\alpha^\omega = \alpha$ then $|X| = nh(X)^{c(X)}\chi(X) = \alpha^\omega = \alpha > 2^\omega$.

Another well-known inequality of Hajnal and Juhász is contained in the next theorem.

**Theorem 3.4** (Hajnal-Juhász). If $X$ is a Hausdorff space then $|X| \leq 2^{c(X)}$.

In Theorem 3.12 we generalize Theorem 3.4 for the class of $T_1$-spaces. In the proof of Theorem 3.12 we will need the following three results (see [6] or [5]):

**Lemma 3.5** (Šapirovskii). Let $\mathcal{U}$ be an open cover of a space $X$ with $s(X) \leq \kappa$. Then there is a subset $A$ of $X$ with $|A| \leq \kappa$ and a subcollection $\mathcal{W}$ of $\mathcal{U}$ with $|\mathcal{W}| \leq \kappa$ such that $X = A \cup (\bigcup \mathcal{W})$.

**Lemma 3.6.** If $X$ is a Hausdorff space then $\psi(X) \leq 2^{c(X)}$. 
Theorem 3.7 (Hajnal-Juhász). If $X$ is a $T_1$-space then $|X| \leq 2^{s(X)\psi(X)}$.

In order to extend Lemma 3.6 to the class of all $T_1$-spaces we need to introduce a new cardinal function.

Definition 3.8. Let $X$ be a space. We define the non-Urysohn number of $X$ with respect to the singletons, $nu_s(X)$, as follows: $nu_s(X) := 1 + \sup\{cl_{\theta}(\{x\}) : x \in X\}$.

Clearly if $X$ is a Hausdorff space then $nu_s(X) = 2$ and for every space $nu_s(X) \leq nu(X)$ and $nu_s(X) \geq nh(X)$ (see Corollary 2.13 and Corollary 2.17).

Lemma 3.9. If $X$ is a $T_1$-space then $\psi(X) \leq nu_s(X) \cdot 2^{s(X)}$.

Proof. Let $s(X) = \kappa$, $nu_s(X) = \tau$ and $x \in X$. Using the fact that $X$ is a $T_1$-space, for each $z \in cl_{\theta}(\{x\})$ we can choose an open neighborhood $U_z$ of $x$ that does not contain $z$. Also, for each $y \notin cl_{\theta}(\{x\})$ we can choose an open set $U_y$ such that $x \notin \overline{U}_y$. Then $\mathcal{U} := \{U_y : y \notin cl_{\theta}(\{x\})\}$ is an open cover of $X \setminus cl_{\theta}(\{x\})$. Therefore, according to Lemma 3.5, there exist subsets $A$ and $B$ of $X \setminus cl_{\theta}(\{x\})$ such that $|A| \leq \kappa$, $|B| \leq \kappa$ and $X \setminus cl_{\theta}(\{x\}) \subseteq A \cup (\bigcup_{y \in B} U_y)$. Let $\mathcal{V} := \{U_z : z \in cl_{\theta}(\{x\})\}$, $\mathcal{V}_A := \{X \setminus \overline{U}_y \cap A : y \in A \setminus \{x\}\}$ and $\mathcal{V}_B := \{X \setminus \overline{U}_y : y \in B\}$. Then $\mathcal{V} \cup \mathcal{V}_A \cup \mathcal{V}_B$ is a pseudobase for $X$ with cardinality $\leq \tau + 2^{\kappa} + \kappa \leq \tau \cdot 2^{\kappa}$. \hfill $\square$

Corollary 3.10. If $X$ is a $T_1$-space then $\psi(X) \leq nu(X) \cdot 2^{s(X)}$.

Remark 3.11. Let $\kappa > 2^\omega$ and $X$ be a space with cardinality $\kappa$, equipped with the cofinite topology. Then $s(X) = \omega$ and $\psi(X) = \kappa$. Hence $\psi(X) = \kappa > 2^\omega = 2^{s(X)}$. Also, for every $x \in X$ we have $cl_{\theta}(\{x\}) = X$. Thus $nu_s(X) = \kappa$. Therefore in the inequality in Lemma 3.9 the cardinal function $nu_s(X)$ cannot be replaced by 2, but we do not know if $nu_s(X)$ cannot be replaced by $nh(X)$. (Note that in our example $nh(X) = \kappa$, as well.)

Now using Lemma 3.9 and Theorem 3.7 we generalize Theorem 3.4 as follows:

Theorem 3.12. If $X$ is a $T_1$-space then $|X| \leq 2^{nu_s(X)\cdot 2^{s(X)}}$.

Proof. $|X| \leq 2^{s(X)\psi(X)} \leq 2^{s(X)\cdot nu_s(X)\cdot 2^{s(X)}} = 2^{nu_s(X)\cdot 2^{s(X)}}$. \hfill $\square$

Corollary 3.13. If $X$ is a $T_1$-space such that $nu_s(X) \leq 2^{s(X)}$ then $|X| \leq 2^{2^{s(X)}}$.

Corollary 3.14. If $X$ is a $T_1$-space then $|X| \leq 2^{nu(X)\cdot 2^{s(X)}}$.

Question 3.15. Is it true that if $X$ is a $T_1$-space then $|X| \leq 2^{nh(X)\cdot 2^{s(X)}}$?
References


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