

PROOF OF LOCKE'S CONJECTURE, I

NELSON CASTAÑEDA AND IVAN S. GOTCHEV

ABSTRACT. Locke's conjecture (L) states that the binary hypercube Q_n with k deleted vertices of each parity is Hamiltonian if $n \geq k + 2$. In 2003, S. C. Locke and R. Stong published in *The American Mathematical Monthly* a proof of (L) for the case $k = 1$. In 2007, in the paper *Path coverings with prescribed ends in faulty hypercubes* the authors proved (L) for every $k \leq 4$ and formulated the following conjecture (CG): Let $n \geq k + 3$ and \mathcal{F} be a set of k even (odd) and $k + 1$ odd (even) vertices of Q_n . If u, v are two even (odd) vertices of $Q_n - \mathcal{F}$ then there exists a Hamiltonian path of $Q_n - \mathcal{F}$ that connects u and v . (CG) is known to be true for every $k \leq 3$ and every $n \geq k + 3$.

In this paper we prove that if $n \geq 7$, $5 \leq k \leq n - 2$ and (L) is true for every $n_1 \leq n - 1$ and every $k_1 \leq n_1 - 2$ and (CG) is true for every $n_1 \leq n - 1$ and every $k_1 \leq n_1 - 3$ then (L) is also true for n and k . To keep the paper shorter the proof that if $n \geq 7$, $4 \leq k \leq n - 3$ and (L) is true for every $n_1 \leq n - 1$ and every $k_1 \leq n_1 - 2$ and (CG) is true for every $n_1 \leq n - 1$ and every $k_1 \leq n_1 - 3$ then (CG) is also true for n and k will appear in a forthcoming paper. In that way the two papers together complete the proofs of (L) and (CG).

1. INTRODUCTION

The n -dimensional binary hypercube Q_n is the graph whose vertices are the binary sequences of length n and whose edges are pairs of binary sequences that differ in exactly one position.

A given vertex is called *even* if it has an even number of 1's in its components; otherwise the vertex is called *odd*.

S. Locke asked the following question in [L]: Let $k \geq 1$ and $n \geq k + 2$ be integers. Let also \mathcal{F} be a set of k even and k odd vertices of Q_n . Is it true that the graph $Q_n - \mathcal{F}$ has a Hamiltonian cycle?

2000 *Mathematics Subject Classification*. Primary 05C38; Secondary 68R10.
Key words and phrases. Hamiltonian path, Hamiltonian cycle, Hypercube.

Since S. Locke had a positive answer to the above question in the case when $k = 1$, we assume that he anticipated a positive answer. Hence, we call the above problem Locke's conjecture and we denote it by (L) .

The Monthly published R. Stong's proof of (L) for the case $k = 1$ and made the remark that Stong had also proved (L) when $n \geq 2k + 3 \log_2 k + 4$ (see [LS]) .

In [CG1] we proved (L) for every $k \leq 4$ and every $n \geq k + 2$, formulated the following conjecture (henceforth denoted by (CG)) and proved it for $k \leq 2$ (the case $k = 0$ had appeared already in [LW]).

Conjecture 1.1. *Let $k \geq 0$ and $n \geq k + 3$ be integers. Let also \mathcal{F} be a set of k even (odd) and $k + 1$ odd (even) vertices of Q_n . If u, v are two even (odd) vertices of $Q_n - \mathcal{F}$ then there exists a Hamiltonian path of $Q_n - \mathcal{F}$ that connects u and v .*

The proof of (CG) for the case $k = 3$ is contained in [CGGL1]. Therefore (L) has been verified for every $k \leq 4$ and every $n \geq k + 2$ and (CG) has been verified for every $k \leq 3$ and every $n \geq k + 3$.

In this paper we prove that if $n \geq 7$, $5 \leq k \leq n - 2$ and (L) is true for every $n_1 \leq n - 1$ and every $k_1 \leq n_1 - 2$ and (CG) is true for every $n_1 \leq n - 1$ and every $k_1 \leq n_1 - 3$ then (L) is also true for n and k . To keep the paper shorter the proof that if $n \geq 7$, $4 \leq k \leq n - 3$ and (L) is true for every $n_1 \leq n - 1$ and every $k_1 \leq n_1 - 2$ and (CG) is true for every $n_1 \leq n - 1$ and every $k_1 \leq n_1 - 3$ then (CG) is also true for n and k will appear in a forthcoming paper [CGGL2]. In that way the two papers together complete the proofs of (L) and (CG) .

In [CG1] the following conjecture (T) was also formulated.

Conjecture 1.2. *Let $k \geq 0$ and $n \geq k + 3$ be integers. Let also \mathcal{F} be a set of k even and k odd vertices of Q_n . If u and v are two vertices of $Q_n - \mathcal{F}$ with different parity then there exists a Hamiltonian path of $Q_n - \mathcal{F}$ that connects u and v .*

For the proof of (L) we need the following theorem which shows that (T) is a simple corollary of (CG) .

Theorem 1.3. *Let $k \geq 0$ and $n \geq k + 3$ be integers. Let also \mathcal{F} be a set of k even and k odd vertices of Q_n and suppose that (CG) is true for n and k . If u and v are two vertices of $Q_n - \mathcal{F}$ with different parity then there exists a Hamiltonian path of $Q_n - \mathcal{F}$ that connects u and v .*

Proof. Let v_1 be a neighbor of u which is not in \mathcal{F} . It follows from (CG) that there exists a Hamiltonian path γ_1 for $Q_n - \mathcal{F} - \{u\}$ from

v_1 to v . Therefore

$$u \longrightarrow v_1 \xrightarrow{\gamma_1} v$$

is a Hamiltonian path for $Q_n - \mathcal{F}$ from u to v . □

2. PRELIMINARIES

To simplify the explanations and the proofs that follow we introduce some terminology.

Every vertex u of Q_n is a binary sequence of length n . We refer to the i -th bit of u as the i -th coordinate of u . All vertices of Q_n with identical i -th coordinates form $(n-1)$ -dimensional hypercubes that we denote by Q_n^0 and Q_n^1 , or Q_n^{bot} and Q_n^{top} , and we call them *bottom and top plates*, respectively. Sometimes we say that *the i -th coordinate splits Q_n into two hypercubes*. If we split the hypercube using two coordinates then we get four $(n-2)$ -dimensional hypercubes that we denote by Q_n^{00} , Q_n^{01} , Q_n^{10} and Q_n^{11} . If u and v are two vertices of Q_n such that $u \in Q_n^0$ and $v \in Q_n^1$ then we say that the i -th coordinate separates u and v . Thus, if \mathcal{F} is any set of vertices of Q_n then each coordinate i induces a partition $\{\mathcal{F}^0, \mathcal{F}^1\}$ of the set \mathcal{F} , where \mathcal{F}^0 and \mathcal{F}^1 is the set of vertices of \mathcal{F} with the i -th coordinate equal to zero or one, respectively. In the special case when one of the sets \mathcal{F}^0 or \mathcal{F}^1 is empty we say that the i -th coordinate does not separate \mathcal{F} . We say that *the coordinate i separates \mathcal{F} in the way (s, t) (or the separation type is (s, t))* if the sets \mathcal{F}^0 and \mathcal{F}^1 are nonempty and have cardinalities s and t . We do not make a difference between the types (s, t) and (t, s) .

We say that *the j -th coordinate separates \mathcal{F} in a different way than the i -th coordinate* if the partitions of \mathcal{F} induced by i and j are different. More generally, we say that a set of k coordinates separates \mathcal{F} in l different ways if the total number of different partitions of \mathcal{F} induced by the k coordinates is l . It is easy to see that if the coordinates i and j separate \mathcal{F} in two different ways then there are vertices from \mathcal{F} in at least three of the four $n-2$ -dimensional hypercubes Q_n^{00} , Q_n^{01} , Q_n^{10} and Q_n^{11} .

Given a vertex a of \mathcal{F} we say that a coordinate is *\mathcal{F} -special for a* if this coordinate separates a from the rest of the vertices of \mathcal{F} .

We view Q_n as the graph $Q_{n-1} \times K_2$, where K_2 is the edge $(0, 1)$, and then $Q_n^0 = Q_{n-1} \times \{0\}$ and $Q_n^1 = Q_{n-1} \times \{1\}$. Also, we view Q_n as the graph $Q_{n-2} \times Q_2$ and then $Q_n^{00} = Q_{n-2} \times \{00\}$, $Q_n^{01} = Q_{n-2} \times \{01\}$, $Q_n^{10} = Q_{n-2} \times \{10\}$ and $Q_n^{11} = Q_{n-2} \times \{11\}$.

For the proof of (L) we need the following separation lemmas.

Lemma 2.1. *Let $k \geq 3$, $n \geq k$, and \mathcal{F} be a set of k pairs of even and odd vertices of Q_n . Let also every coordinate that separates the even vertices in \mathcal{F} separates also the odd vertices in \mathcal{F} and vice versa. Then there exist two coordinates that separate the even and the odd vertices in \mathcal{F} in different ways.*

Proof. Take one coordinate, say A , that separates the even and the odd vertices in \mathcal{F} . Since there are at least three odd vertices in \mathcal{F} then at least two of them are not separated by A . Take another coordinate, say B , that separates these two odd vertices. Then B separates the odd vertices in \mathcal{F} in a different way than A . If B also separates the even vertices in \mathcal{F} in a different way than A then A and B are as required.

Suppose that B separates the even vertices in \mathcal{F} in the same way as A . Since there are at least three even vertices in \mathcal{F} at least two of them are not separated by A (and by B). Let C be a coordinate that separates these two even vertices. Clearly, C separates the even vertices in \mathcal{F} in a different way than A and B . Also, C separates the odd vertices in \mathcal{F} . If C separates the odd vertices in \mathcal{F} in a different way than A then A and C are as required, otherwise C and B are the required two coordinates. \square

Lemma 2.2. *Let $k \geq 3$, $n \geq k$, and \mathcal{F} be a set of k even (odd) vertices of Q_n . If every coordinate that separates the vertices in \mathcal{F} separates them in the way $(1, k-1)$ then either there exist k coordinates that separate all vertices in \mathcal{F} in k different ways or there exist $2k-2$ coordinates that separate all vertices in \mathcal{F} in $k-1$ different ways. These $2k-2$ coordinates can be arranged in $k-1$ pairs of coordinates such that the two coordinates of each pair separate \mathcal{F} in the same way.*

Proof. It is clear that if $|\mathcal{F}| \geq 3$ then no two vertices can have a common \mathcal{F} -special coordinate. If for every vertex in \mathcal{F} there is an \mathcal{F} -special coordinate then there are at least k coordinates that separate the vertices of \mathcal{F} in k different ways.

Assume now that there is a vertex a in \mathcal{F} with no \mathcal{F} -special coordinates. If b is any other vertex in \mathcal{F} then it differs from a in at least two coordinates and these two coordinates must by necessity be \mathcal{F} -special for b . Therefore we have $k-1$ pairs of coordinates with the properties stated in the lemma. \square

Lemma 2.3. *Let $n \geq 3$ be a positive integer and \mathcal{F} be a set of four odd (even) vertices of Q_n . Then there exist three coordinates that separate*

these vertices in three different ways or there are two pairs of coordinates such that these coordinates separate \mathcal{F} in two different ways with the two coordinates of each pair separating \mathcal{F} in the same way that is of type $(2, 2)$.

Proof. Let $\mathcal{F} = \{a, b, c, d\}$. If at least one vertex in \mathcal{F} , say a , has a coordinate that is \mathcal{F} -special for a then this coordinate together with any two coordinates that separate $\{b, c, d\}$ in two different ways (which exists by Lemma 2.2) form a group of three coordinates that separate \mathcal{F} in three different ways. So, if \mathcal{F} cannot be separated in three different ways then all the separations of \mathcal{F} are of the type $(2, 2)$. Also, without loss of generality, we can assume that d has no $\{b, c, d\}$ -special coordinates and that one pair of coordinates separates b from $\{c, d\}$ and another pair of coordinates separates c from $\{b, d\}$ (see Lemma 2.2). It follows that the first pair of coordinates separate \mathcal{F} as $\{a, b\}, \{c, d\}$ and the second pair of coordinates separates \mathcal{F} as $\{a, c\}, \{b, d\}$. \square

Lemma 2.4. *Let $n \geq 4$ be a positive integer and $\mathcal{F} = \{a, b, c, d, e\}$ be a set of five odd (even) vertices of Q_n . Then there exist four coordinates that separate these vertices in four different ways or there exist three pairs of coordinates that separate \mathcal{F} in three different ways with the two coordinates of each pair separating \mathcal{F} in the same way that is of type $(2, 3)$.*

Proof. There are two types of separations for a set of 5 vertices: $(1, 4)$, and $(2, 3)$. If all the separations of \mathcal{F} are of type $(1, 4)$ then according to Lemma 2.2 there are at least four coordinates that separate \mathcal{F} in four different ways. Now, suppose that there are separations of \mathcal{F} of type $(2, 3)$. Without loss of generality we can assume that $\Pi = \{\{a, b\}, \{c, d, e\}\}$ is a separation of \mathcal{F} produced by some coordinate A . If c, d, e can be separated in three different ways by three coordinates then these three coordinates together with A form a group of four coordinates that separate \mathcal{F} in four different ways. If c, d, e cannot be separated in three different ways then, according to Lemma 2.2 and without loss of generality, we can assume that there are two coordinates, say C and D , that separate c, d, e in the way $\{c\}, \{d, e\}$ and two coordinates, say E and F , that separate c, d, e in the way $\{d\}, \{c, e\}$. If C (E) separates \mathcal{F} in a different way than D (F) then A, C, D, E (A, C, E, F) form a group of four coordinates that separate \mathcal{F} in four different ways. Assume now that C separates \mathcal{F} in the same way as D and that E separates \mathcal{F} in the same way as F . If neither C nor E separates a from b then any coordinate that separates a from b together

with A, C and E form a group of four coordinates that separate \mathcal{F} in four different ways. Assume now that one of the coordinates C or E separates a from b . Without loss of generality we can assume that that coordinate is C and that C separates \mathcal{F} in the way $\{a, c\}, \{b, d, e\}$. If any of the coordinates that separate b, d, e separates \mathcal{F} in a different way than A, C , and E then such coordinate together with A, C and E form a group of four coordinates that separate \mathcal{F} in four different ways. If that is not the case then b, d, e can be separated only in the ways produced by A, C and E . In particular (by Lemma 2.2) there must be a coordinate B that separates \mathcal{F} in exactly the same way as A does. \square

Corollary 2.5. *Let $k \geq 5$, $n = k + 2$, and \mathcal{F} be a set of k even and k odd vertices of Q_n . If every coordinate that separates the even (odd) vertices in \mathcal{F} separates them in the way $(1, k - 1)$ then there exist two coordinates that separate the even and the odd vertices in different ways.*

Proof. Since every coordinate that separates the even vertices separates them in the way $(1, k - 1)$, it follows from Lemma 2.2 that there are either k coordinates that separate the even vertices in different ways or $2k - 2$ coordinates that separate the even vertices in $k - 1$ different ways. Since $k \geq 5$, it follows from Lemma 2.4 that either there exist at least four coordinates that separate the odd vertices in different ways or there exist three pairs of coordinates that separate the odd vertices in three different ways with the two coordinates from each pair separating the odd vertices in the same way. In either case, since $n = k + 2$, there will be two coordinates that separate the even and the odd vertices in different ways. \square

An important ingredient in the proofs of (L) and (CG) is the existence of a long enough path that avoids a set of faulty vertices as the one guaranteed by Lemma 2.7 below. For the proof of Lemma 2.7 we need the following result.

Theorem 2.6 ([CG2]). *Let $n \geq 5$ and f be integers with $0 \leq f \leq 3n - 7$. Then for any set \mathcal{F} of vertices of Q_n of cardinality f there exists a cycle in $Q_n - \mathcal{F}$ of length at least $2^n - 2f$.*

Lemma 2.7. *Let $n \geq 5$ be an integer and \mathcal{F} be a set of $2n$ vertices of Q_n . Then there exists a path γ in $Q_n - \mathcal{F}$ with length at least $2(n-3)+2$.*

Proof. We have $|\mathcal{F}| = 2n$.

If $n \geq 7$ then $3n - 7 \geq 2n$. Therefore, according to Theorem 2.6, there is a Hamiltonian cycle in $Q_n - \mathcal{F}$ with length at least $2^n - 2(2n)$. Since

$2^n - 2(2n) \geq 2(n - 3) + 3$, when $n \geq 5$, we conclude that if $n \geq 7$ there is a path γ in $Q_n - \mathcal{F}$ with length at least $2(n - 3) + 2$.

If $n = 6$ then it follows from Theorem 2.6 that we can find a cycle in Q_n with length at least $2^6 - 2 \cdot 11 = 42$, such that this cycle contains at most one of the vertices from \mathcal{F} , for in this case $|\mathcal{F}| \leq 12$. Therefore, when $n = 6$ there exists a path γ in $Q_n - \mathcal{F}$ with length at least $2(n - 3) + 2 = 8 \leq 40$.

Finally, if $n = 5$, again using Theorem 2.6, we can find a cycle in Q_n with length at least $2^5 - 2 \cdot 8 = 16$ that contains at most two of the vertices from \mathcal{F} . Therefore, when $n = 5$, there exists a path γ in $Q_n - \mathcal{F}$ with length at least $6 = 2(n - 3) + 2$. \square

As a corollary of Lemma 2.7 we obtain the following very useful lemma.

Lemma 2.8. *Let $k \geq 1$ and $n \geq 7$ be integers, with $n \geq k + 2$, and \mathcal{F} be a set of k even and k odd vertices of Q_n . Split Q_n using two coordinates and let Q_{n-2} be one of the four hypercubes Q_n^{00} , Q_n^{01} , Q_n^{10} or Q_n^{11} . Project all vertices from \mathcal{F} onto Q_{n-2} using the natural projections and denote the projection by \mathcal{F}' . Then there exists a path μ in $Q_{n-2} - \mathcal{F}'$ with length $2(n - 5) + 1$. Since the length of μ is odd we can choose the beginning vertex of μ to be either even or odd depending on our needs.*

Proof. Since $|\mathcal{F}| \leq 2k \leq 2(n - 2)$, we have $|\mathcal{F}'| \leq 2(n - 2)$. Also $n - 2 \geq 5$. Therefore, it follows from Lemma 2.7 that there exists a path γ in $Q_{n-2} - \mathcal{F}'$ with length at least $2(n - 5) + 2$ and therefore there exists a path μ with length $2(n - 5) + 1$ that begins with an even or odd vertex, depending on our choice. \square

3. PROOF OF LOCKE'S CONJECTURE

In this section we complete the prove of Locke's conjecture. For that end we prove the following: Let $n \geq 7$ and $5 \leq k \leq n - 2$ be integers. Let also \mathcal{F} be a set of k even and k odd vertices of Q_n and suppose that (L) is true for every $n_1 \leq n - 1$ and every $k_1 \leq n_1 - 2$ and (CG) is true for every $n_1 \leq n - 1$ and every $k_1 \leq n_1 - 3$. Then (L) is also true for n and k . Notice that under these hypotheses it follows from Theorem 1.3 that (T) is true for every $n_1 \leq n - 1$ and every $k_1 \leq n_1 - 3$, as well.

Let $\mathcal{F} = \{u_1, \dots, u_k, v_1, \dots, v_k\}$, where all u -s are even and all v -s are odd vertices. Sometimes we call the elements of \mathcal{F} *deleted* vertices.

The idea of the proof is to choose “appropriately” one or two coordinates that separate the deleted vertices “in a good way” and to split \mathcal{Q}_n using them. Then using (L) , (CG) or (T) for some $n_1 \leq n - 1$ and $k_1 \leq k - 1$ to construct the required Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$. In many cases it is impossible to find one coordinate such that immediately after the splitting we can use (L) , (CG) or (T) in the resulting hypercubes \mathcal{Q}_n^{top} and \mathcal{Q}_n^{bot} since usually there is a big difference (more than one) or *disbalance* between the number of the deleted even and odd vertices in these hypercubes. In such cases we choose “appropriately” two coordinates and using them we split \mathcal{Q}_n into four hypercubes \mathcal{Q}_n^{00} , \mathcal{Q}_n^{01} , \mathcal{Q}_n^{10} and \mathcal{Q}_n^{11} . Then, we start creating a path γ_0 that is the first part of the desired cycle of $\mathcal{Q}_n - \mathcal{F}$ by concatenating paths of the type

$$e_{01} \rightarrow o_{00} \rightarrow (e, o)_{10} \rightarrow e_{11} \rightarrow (o, e')_{01}$$

or other similar types that we call *short cycles* (since the projection of each such path on \mathcal{Q}_2 is a cycle). We refer to all vertices used in those short cycles, except the starting one as *used* vertices. We stop the creation of γ_0 at a point when the set of deleted or used vertices in each of the four hypercubes is *balanced* or *semi-balanced* in the sense that the disbalance between even and odd vertices is at most one. Considering the originally deleted vertices and the now used vertices as new deleted vertices we proceed by creating paths in each of the four hypercubes, applying (L) , (CG) or (T) as needed, to complete the desired cycle of $\mathcal{Q}_n - \mathcal{F}$.

The notation that we use in these short cycles is self-explanatory: e_{01} represents an even (in \mathcal{Q}_n) vertex which is in the hypercube \mathcal{Q}_n^{01} ; o_{00} represents an odd (in \mathcal{Q}_n) vertex which is in the hypercube \mathcal{Q}_n^{00} and is a neighbor of e_{01} in \mathcal{Q}_n ; $e_{01} \rightarrow o_{00}$ means that (e_{01}, o_{00}) is an edge in \mathcal{Q}_n and an edge in the constructed path; $(e, o)_{10}$ represents the edge (e_{10}, o_{10}) in the hypercube \mathcal{Q}_n^{10} which is also an edge in the constructed path; $o_{00} \rightarrow (e, o)_{10}$ means that o_{00} and e_{10} are neighbors in \mathcal{Q}_n and that (o_{00}, e_{10}) is an edge in the constructed path; and so on. We call the edges of the type $(e_{ij}, o_{i_1j_1})$, where $ij \neq i_1j_1$, *vertical*, and the edges of the type (e_{ij}, o_{ij}) *horizontal*.

Usually more than one short cycle is needed in order to (semi) balance the four plates. Since each one of those short cycles will be part of the required Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we do not want different short cycles to use the same vertices and to contain deleted vertices. In order to guarantee that, in the beginning of each construction we project all deleted vertices on one of the four hypercubes \mathcal{Q}_n^{00} , \mathcal{Q}_n^{01} , \mathcal{Q}_n^{10}

and \mathcal{Q}_n^{11} , where the construction of the Hamiltonian cycle begins, and using Lemma 2.8 we choose a path μ in that hypercube with length at least $2(n - 5) + 1$ that begins with an even or an odd vertex (in \mathcal{Q}_n), depending on our needs. Using the natural projections we identify all four hypercubes \mathcal{Q}_n^{00} , \mathcal{Q}_n^{01} , \mathcal{Q}_n^{10} and \mathcal{Q}_n^{11} and in that way we obtain four copies of μ : μ_{00} , μ_{01} , μ_{10} and μ_{11} – one in each of the four hypercubes. Then, to construct the short cycles, we follow μ , i.e. every vertex from each short cycle which is in \mathcal{Q}_n^{ii} belongs to μ_{ii} and every horizontal edge which is in \mathcal{Q}_n^{ii} belongs to μ_{ii} . In each short cycle we use at least one and at most two horizontal edges, hence for each short cycle the first and the last vertex are different and for each short cycle we use at most two edges from μ . Also, we always traverse μ in the same direction and therefore we never use the same edge from μ twice. Therefore, at the end of the construction, our short cycles do not contain deleted vertices and every undeleted vertex is contained in at most one short cycle. Clearly the length of μ is enough to construct at least $n - 5 \geq k - 3$ such short cycles.

In the proofs below we refer to the path μ described above as a *model path* and shall not repeat each time how we choose μ when we use it. Also, whenever we construct short cycles in the proofs below we shall use the model path μ and the procedure described above without mentioning that specifically.

In order to explain how we choose the coordinates that we use to split \mathcal{Q}_n we order all vertices from \mathcal{F} in a column and let M' be the $2k \times n$ matrix determined by the coordinates of those vertices (every row corresponds to a vertex). Then every coordinate corresponds to a column in M' and every column in M' corresponds to a coordinate, hence we shall not make a difference between columns and coordinates. Let M_e be the submatrix of M' determined by those columns in M' that separate only the even vertices in \mathcal{F} and M_o be the submatrix of M' determined by those columns in M' that separate only the odd vertices in \mathcal{F} . For two disjoint submatrices A and B of M' by (A, B) we denote the submatrix of M' determined by the columns that are in A or in B . (The order of the columns in all matrices that we consider here is not important to us). let also M_2 be the submatrix of M' determined by those columns in M' that separate simultaneously the even and the odd vertices in \mathcal{F} . Finally, set $M_1 = (M_e, M_o)$ and $M = (M_1, M_2)$.

Now we shall show that we can always choose one or two columns from M that satisfy at least one of the cases (A) – (J) considered below and therefore proving that all those cases completes the proof of (L).

In cases (A) – (C), the existence of one column in M that separates the deleted vertices in a special way is sufficient for the construction of the required Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$: the case when there exists a column in M_1 that separates the vertices in \mathcal{F} in the way $(1, 2k - 1)$ is considered in (A); the case when there exists a column in M_1 that separates the vertices in \mathcal{F} in the way $(2, 2k - 2)$ is considered in (B); and the case when there is a column in M_2 that separates the odd vertices in the way $(s, k - s)$, the even vertices in the way $(s, k - s)$, and all vertices in \mathcal{F} in the way $(2s, 2k - 2s)$, where $1 \leq s \leq k - 1$, is considered in (C).

Remark 3.1. *In the remaining cases (D) – (J), without loss of generality, we assume that there are no columns in M that fall in any of the cases (A) – (C). In particular, we assume that every column that separates only the odd or only the even vertices in the way $(1, k - 1)$ separates all vertices in \mathcal{F} in the way $(k + 1, k - 1)$ and every column that separates only the odd or only the even vertices in the way $(2, k - 2)$ separates all vertices in \mathcal{F} in the way $(k + 2, k - 2)$.*

In the remaining cases (D) – (J) two columns are required for the construction of the Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$.

The case when there exist two columns in M_2 that separate the even and the odd vertices in different ways is considered in (D).

Remark 3.2. *In the remaining cases (E) – (J) we assume that columns as in (D) do not exist.*

For cases (E) – (G) we assume that there exists a column A in M_2 that separates the even vertices in the way $(r, k - r)$, where $2 \leq r \leq k - 2$.

Since $k > 2$, there is a column B in M that separates the odd vertices in a different way than A . The case when B is in M_2 and separates the even vertices in the same way as A is considered in (E). Now, assume that there is no such column in M_2 , hence every column in M_2 separates the odd vertices as A does. Therefore B is in M_o . The case when A or B separates the odd vertices in the way $(s, k - s)$, where $2 \leq s \leq k - 2$, is considered in (F). If neither A nor any B from M_o separates the odd vertices in the way $(s, k - s)$, where $2 \leq s \leq k - 2$, then every column that separates the odd vertices separates them in the way $(1, k - 1)$ ¹. Then it follows from Corollary 2.5 that if $k = n - 2$ then there are two columns in M_2 that separate the even and the odd

¹Recall that we are assuming that no two columns that separate both the even and the odd vertices in two different ways exist.

vertices in different ways, which is case (D) . Therefore we can assume that $k \leq n - 3$. This case is considered in (G) .

For the remaining cases $(H) - (J)$ we assume that every column in M_2 separates the even and the odd vertices in the way $(1, k - 1)$.

The case when there is a column in M_2 that separates all vertices in \mathcal{F} in the way $(2, 2k - 2)$ was considered in (C) . Therefore we can assume that every column in M_2 separates the vertices in \mathcal{F} in the way (k, k) . If M_1 is empty, or equivalently, $M = M_2$, then according to Lemma 2.1, there exist two columns in M that separate the even and the odd vertices in different ways, which is impossible according to Remark 3.2. Hence, we can assume that M_1 is non-empty and therefore, without loss of generality, we can assume that M_o is non-empty.

Let A be a column in M_o . Suppose that A separates the odd vertices in the way $(s, k - s)$, where $2 \leq s \leq k - 2$. The case when there exists a column in M_e that separates the even vertices in the way $(r, k - r)$, where $2 \leq r \leq k - 2$, is considered in (H) .

Assume now that every column that separates the odd vertices in \mathcal{F} separates them in the way $(1, k - 1)$. Then it follows from Corollary 2.5 that if $k = n - 2$ then there are two columns in M_2 that separate the even and the odd vertices in different ways, which is impossible according to Remark 3.2. Therefore we can assume that $k \leq n - 3$ and that there are no two columns in M_2 that separate the even and the odd vertices in different ways. Then there is a column B that either separates only the even vertices or if B separates also the odd vertices then B separates them in a different way than A . These cases are considered in (I) and (J) and that completes the list of the cases that need to be considered in order to prove (L) .

Now we begin considering case by case.

Case (A) There is a column A in M_1 that separates the vertices in \mathcal{F} in the way $(1, 2k - 1)$.

Use A to split the hypercube. Without loss of generality we can assume that there are k even and $k - 1$ odd deleted vertices in the top plate and one odd vertex in the bottom plate. Use (CG) for $n - 1$ and $k - 1$ to find a Hamiltonian cycle for the top plate that contains one of the deleted odd vertices. Then delete that odd vertex from the cycle and connect the resulting path to the bottom plate with two edges that we call bridges. Then use (CG) for $n - 1$ and 0 to find a Hamiltonian path for the bottom plate that connects the end vertices of the bridges

and does not contain the deleted odd vertex. The result is the desired Hamiltonian cycle.

Case (B) There exists a column A in M_1 that separates the vertices in \mathcal{F} in the way $(2, 2k - 2)$.

Without loss of generality we can assume that A belongs to M_o . Thus, if we split \mathcal{Q}_n using A , there will be two odd vertices in one of the plates, say the top plate, and $2k - 2$ deleted vertices in the bottom plate. Since there are at least five deleted even vertices in the bottom plate there are two, say e_1 and e_2 , that are at distance at least four. Use (L) for $n - 1$ and $k - 2$ to find a Hamiltonian cycle γ for the bottom plate that contains e_1 and e_2 and avoids all the other $k - 2$ pairs of deleted even and odd vertices. Delete e_1 and e_2 from γ . In that way we obtain a 2-path covering for the bottom plate that does not contain any of the deleted vertices. Connect the end vertices of both paths with bridges to the top plate. Use [CG1, Lemma 4.3] to find a 2-path covering of the top plate that avoids the two deleted odd vertices, each path connects two end vertices of two of the bridges, and such that these two paths together with the bridges and the other two paths form the desired Hamiltonian cycle.

Case (C) There exists a column A in M_2 that separates the odd vertices in the way $(s, k - s)$, the even vertices in the way $(s, k - s)$, and all vertices in \mathcal{F} in the way $(2s, 2k - 2s)$, where $1 \leq s \leq k - 1$.

Without loss of generality we can assume that $s \leq k - s$. Since $k \geq 5$, s and $k - s$ cannot be simultaneously equal to $k - 1$. It follows from our hypothesis that if we split the hypercube using A , there will be $k - s \leq k - 1 \leq (n - 1) - 2$ pairs of deleted even and odd vertices in one of the plates and $s \leq k - 2 \leq (n - 2) - 2 = (n - 1) - 3$ pairs of deleted even and odd vertices in the other plate. Use (L) for $n - 1$ and $k - s$ to find a Hamiltonian cycle for the plate that contains $2k - 2s$ deleted vertices that avoids all the deleted vertices. Cut that cycle at an edge whose end vertices are not neighbors of any of the deleted vertices on the other plate. Such edge exists since the length of the Hamiltonian cycle is $2^{n-1} - 2(k - s) > 4s$ and there are only $2s$ deleted vertices on the other plate. Connect the ends of the resulting path with bridges with the other plate. Use (T) for $n - 1$ and s to find a Hamiltonian path for the plate that contains $s \leq (n - 1) - 3$ deleted pairs of vertices that connects the end vertices of both bridges and avoids all deleted vertices. The result is the desired Hamiltonian cycle.

Note 1. For the remaining cases Remark 3.1 applies.

Case (D) There exist two columns A and B in M_2 that separate the odd and the even vertices in different ways.

We split the hypercube using A and B into the four plates \mathcal{Q}_n^{00} , \mathcal{Q}_n^{01} , \mathcal{Q}_n^{10} , and \mathcal{Q}_n^{11} . Then there will be deleted even (odd) vertices in at least three of the plates, hence the maximal number of deleted even or odd vertices in a given plate could be at most $k - 2$. Also, there will be deleted even and odd vertices in at least two of the plates, hence there will be at least two pairs of even and odd vertices such that each one is contained in one of the four plates.

For easier explanation, for a hypercube K , we use the following terminology: if there are s deleted even and t deleted odd vertices in K then $|s - t|$ is called *charge* of K ; when $s - t > 0$ we say that K has a *positive charge*; when $s - t < 0$ we say that K has a *negative charge*; and when $s - t = 0$ we say that K is *neutral*.

If there exist two plates at distance one which union is a neutral hypercube then that case was considered in (C). Therefore, without loss of generality, we can make the following assumption:

Assumption. Every hypercube, which is the union of two of the four plates which are at distance one, is not neutral.

It follows from Assumption that there exists at least one plate K_1 with a positive charge and at least one plate K_2 with a negative charge. Let the charge of K_1 be $s > 0$ and the charge of K_2 be $t > 0$. We denote by q_{ij} the maximal number of pairs of deleted even and odd vertices that can be formed in the plate \mathcal{Q}_n^{ij} .

We consider two cases: (1) The plates K_1 and K_2 are at distance two; and (2) K_1 and K_2 are at distance one from each other.

(1) K_1 and K_2 are at distance two.

Without loss of generality we can assume that $K_1 = \mathcal{Q}_n^{00}$ and $K_2 = \mathcal{Q}_n^{11}$. Then, up to symmetry and up to interchanging positive and negative charge, there are four different cases: (a) \mathcal{Q}_n^{01} and \mathcal{Q}_n^{10} are neutral; (b) \mathcal{Q}_n^{10} has a negative charge and \mathcal{Q}_n^{01} is neutral; (c) \mathcal{Q}_n^{01} and \mathcal{Q}_n^{10} have negative charges; and (d) \mathcal{Q}_n^{01} has a positive charge and \mathcal{Q}_n^{10} has a negative charge.

(a) \mathcal{Q}_n^{01} and \mathcal{Q}_n^{10} are neutral.

Since there are even (odd) deleted vertices in at least three of the four plates, we have $s = t \leq k - 2$, $q_{01} + t \leq k - 1$, $q_{00} + s \leq k - 2$, $q_{10} + s \leq k - 1$ and $q_{11} + t \leq k - 2$.

Take a model path μ in \mathcal{Q}_n^{01} that begins with an even vertex $e_{01} = u_{01}$ and following μ make $s - 1$ short cycles of the type

$$e_{01} \rightarrow o_{00} \rightarrow (e, o)_{10} \rightarrow e_{11} \rightarrow (o, e')_{01}.$$

We denote the resulting path by γ_0 , its end vertex by a_{01} and let the odd neighbor of a_{01} in \mathcal{Q}_n^{00} be v_{00} . We extend the constructed path so far with the edge (a_{01}, v_{00}) .

The total number of the constructed short cycles is $s - 1 \leq k - 3$, hence the length of μ is enough for that construction.

Let v'_{00} be any unused and undeleted odd vertex in \mathcal{Q}_n^{00} different from v_{00} whose even neighbor u_{10} in \mathcal{Q}_n^{10} is neither a deleted nor used vertex. There are $q_{00} + s$ deleted or used even and $q_{00} + s - 1$ deleted or used odd vertices in \mathcal{Q}_n^{00} . Since $q_{00} + s - 1 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{00} minus all deleted and used vertices that connects v_{00} to v'_{00} .

To continue we need to construct paths γ_2 and γ_4 in \mathcal{Q}_n^{10} and \mathcal{Q}_n^{01} , respectively. For that end we consider two cases: (i) there exists a deleted even vertex in \mathcal{Q}_n^{11} , hence $q_{11} > 0$; and (ii) there are no deleted even vertices in \mathcal{Q}_n^{11} , hence $q_{11} = 0$.

(i) $q_{11} > 0$.

In this case we have $s = t \leq k - 3$ and $q_{01} + q_{11} + t \leq k - 1$, hence $q_{01} + t \leq k - 2$. Notice also that $q_{10} + q_{11} + t \leq k - 1$, hence $q_{10} + t \leq k - 2$.

There are $q_{10} + s - 1 = q_{10} + t - 1$ even and odd used or deleted vertices in \mathcal{Q}_n^{10} (u_{10} is not counted). Let v_{10} be any unused and undeleted odd vertex in \mathcal{Q}_n^{10} whose even neighbor u_{11} in \mathcal{Q}_n^{11} is neither a deleted nor used vertex. Since $q_{10} + t - 1 \leq k - 3 \leq (n - 2) - 3$, it follows from (T) that there exists a Hamiltonian path γ_2 for \mathcal{Q}_n^{10} minus all deleted and used vertices that connects u_{10} to v_{10} .

There are $q_{01} + s - 1 = q_{01} + t - 1$ deleted or used even and odd vertices in \mathcal{Q}_n^{01} (u_{01} is not counted). Let v_{01} be any unused and undeleted odd vertex in \mathcal{Q}_n^{01} which even neighbor u'_{11} in \mathcal{Q}_n^{11} is different from u_{11} and is neither a deleted nor used vertex. Since $q_{01} + t - 1 \leq k - 3 \leq (n - 2) - 3$, it follows from (T) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted and used vertices that connects v_{01} to u_{01} .

(ii) $q_{11} = 0$.

There are $q_{10} + s - 1 = q_{10} + t - 1$ even and odd used or deleted vertices in \mathcal{Q}_n^{10} (u_{10} is not counted). Since $q_{10} + s - 1 \leq k - 2 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{10} minus

all deleted or used vertices. This cycle contains u_{10} . Let v_{10} be an odd neighbor of u_{10} in γ' . Since $q_{11} = 0$, the even neighbor u_{11} of v_{10} in \mathcal{Q}_n^{11} is neither a deleted nor used vertex. We denote by γ_1 the Hamiltonian path for \mathcal{Q}_n^{10} minus all deleted or used vertices that is defined by γ' and connects u_{10} to v_{10} .

There are $q_{01} + s - 1$ deleted or used even and odd vertices in \mathcal{Q}_n^{01} (u_{01} is not counted). Since $q_{01} + s - 1 \leq k - 2 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ'' for \mathcal{Q}_n^{01} minus all deleted or used vertices. This cycle contains u_{01} . Let v_{01} be an odd neighbor of u_{01} in γ'' such that its even neighbor u'_{11} of v_{01} in \mathcal{Q}_n^{11} is different from u_{11} . Since $q_{11} = 0$, u'_{11} is neither a deleted nor used vertex. We denote by γ_4 the Hamiltonian path for \mathcal{Q}_n^{01} minus all deleted or used vertices that is defined by γ'' and connects v_{01} to u_{01} .

Now we continue the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ for both cases (i) and (ii).

There are $q_{11} + t - 1$ deleted or used even and $q_{11} + t$ deleted or used odd vertices in \mathcal{Q}_n^{11} . Since $q_{11} + t - 1 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{11} minus all deleted and used vertices that connects u_{11} to u'_{11} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path γ_0 with the path

$$v_{00} \xrightarrow{\gamma_1} v'_{00} \rightarrow u_{10} \xrightarrow{\gamma_2} v_{10} \rightarrow u_{11} \xrightarrow{\gamma_3} u'_{11} \rightarrow v_{01} \xrightarrow{\gamma_4} u_{01}.$$

(b) \mathcal{Q}_n^{10} has a negative charge and \mathcal{Q}_n^{01} is neutral.

Let the charge of \mathcal{Q}_n^{10} be $p > 0$, hence $p+t = s \leq k-2$ and $s+q_{01} \leq k-1$. Notice also that $q_{00} + s \leq k-2$, $q_{10} + p + t \leq k-1$ and $q_{11} + p + t \leq k-1$ since there are even (odd) deleted vertices in at least three of the four plates.

Take a model path μ in \mathcal{Q}_n^{11} that begins with an even vertex $e_{11} = u_{11}$ and following μ make $p - 1$ short cycles of the type

$$e_{11} \rightarrow (o, e)_{01} \rightarrow o_{00} \rightarrow e_{10} \rightarrow (o, e')_{11}.$$

We denote the end vertex of the resulting path by a_{11} .

Then begin with $e_{11} = a_{11}$ and following μ make $t - 1$ short cycles of the type

$$e_{11} \rightarrow (o, e)_{01} \rightarrow o_{00} \rightarrow (e, o)_{10} \rightarrow e'_{11}.$$

We denote the end vertex of the resulting path by a'_{11} .

Finally, begin with $e_{11} = a'_{11}$ and following μ make the following path with length four

$$e_{11} \rightarrow (o, e)_{01} \rightarrow o_{00} \rightarrow e_{10}.$$

We denote the resulting path by γ_0 and its end vertex by u_{10} .

The total number of the constructed short cycles is $p - 1 + t - 1 + 1 = p + t - 1 \leq k - 3$, hence the length of μ is enough for that construction.

There are $q_{01} + p - 1 + t - 1 + 1 = q_{01} + s - 1 \leq k - 2$ even and odd deleted or used vertices in \mathcal{Q}_n^{01} . Since $k - 2 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{01} minus all deleted or used vertices. Let (u_{01}, v_{01}) be any edge in γ' such that the even neighbor u'_{11} of v_{01} in \mathcal{Q}_n^{11} and the odd neighbor v_{00} of u_{01} in \mathcal{Q}_n^{00} are neither deleted nor used vertices. We denote by γ_3 the Hamiltonian path for \mathcal{Q}_n^{01} minus all deleted and used vertices that is defined by γ' and connects u_{01} to v_{01} .

Let v'_{00} be any unused and undeleted odd vertex in \mathcal{Q}_n^{00} different from v_{00} which even neighbor u'_{10} in \mathcal{Q}_n^{10} is neither deleted nor used vertex. There are $q_{00} + s$ used or deleted even and $q_{00} + p - 1 + t - 1 + 1 = q_{00} + s - 1$ used or deleted odd vertices in \mathcal{Q}_n^{00} . Since $q_{00} + s - 1 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_2 for \mathcal{Q}_n^{00} minus all deleted and used vertices that connects v'_{00} to v_{00} .

There are $q_{10} + p - 1 + t - 1 = q_{10} + p + t - 2$ used or deleted even and $q_{10} + p - 1 + t - 1 + 1 = q_{10} + p + t - 1$ deleted or used odd vertices in \mathcal{Q}_n^{10} (u_{10} and u'_{10} are not counted). Since $q_{10} + p + t - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{10} minus all deleted and used vertices that connects u_{10} to u'_{10} .

There are $q_{11} + p - 1 + t - 1 = q_{11} + p + t - 2$ deleted or used even and $q_{11} + p - 1 + t = q_{11} + p + t - 1$ deleted or used odd vertices in \mathcal{Q}_n^{11} . Since $q_{11} + p + t - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{11} minus all deleted and used vertices that connects u'_{11} to u_{11} .

Finally, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \{u_1, \dots, u_k, v_1, \dots, v_k\}$ we extend the previously constructed path γ_0 with the path

$$u_{10} \xrightarrow{\gamma_1} u'_{10} \rightarrow v'_{00} \xrightarrow{\gamma_2} v_{00} \rightarrow u_{01} \xrightarrow{\gamma_3} v_{01} \rightarrow u'_{11} \xrightarrow{\gamma_4} u_{11}.$$

(c) \mathcal{Q}_n^{01} and \mathcal{Q}_n^{10} have negative charges.

Let the charge of \mathcal{Q}_n^{01} be p and the charge of \mathcal{Q}_n^{10} be r . Then $p + t + r = s \leq k - 2$ and $s + q_{11} \leq k - 1$. Notice also that $q_{00} + s \leq k - 2$,

$q_{10} + s \leq k - 1$ and $q_{01} + s \leq k - 1$ since there are even (odd) deleted vertices in at least three of the four plates.

Take a model path μ in \mathcal{Q}_n^{01} that begins with an even vertex $e_{01} = u_{01}$ and following μ make $p - 1$ short cycles of the type

$$e_{01} \rightarrow o_{00} \rightarrow (e, o)_{10} \rightarrow (e, o)_{11} \rightarrow e'_{01}.$$

We denote the end vertex of the resulting path by a_{01} .

Then begin with $e_{01} = a_{01}$ and following μ make $r - 1$ short cycles of the type

$$e_{01} \rightarrow o_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow (o, e')_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Now begin with $e_{01} = a'_{01}$ and following μ make t short cycles of the type

$$e_{01} \rightarrow o_{00} \rightarrow (e, o)_{10} \rightarrow e_{11} \rightarrow (o, e')_{01}.$$

We denote the end vertex of the resulting path by a''_{01} .

Finally, begin with $e_{01} = a''_{01}$ and following μ make the following path with length three

$$e_{01} \rightarrow o_{00} \rightarrow e_{10} \rightarrow o_{11}.$$

We denote the end vertex of the resulting path by v_{11} .

The total number of the constructed short cycles is $p - 1 + r - 1 + t = s - 2 \leq k - 4$, hence the length of μ is enough for that construction.

There are $q_{10} + p - 1 + t + r - 1 + 1 = q_{10} + s - 1 \leq k - 2$ even and odd deleted or used vertices in \mathcal{Q}_n^{10} . Since $k - 2 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{10} minus all deleted or used vertices. Let (u_{10}, v_{10}) be any edge in γ' such that the even neighbor u_{11} of v_{10} in \mathcal{Q}_n^{11} and the odd neighbor v_{00} of u_{10} in \mathcal{Q}_n^{00} are neither deleted nor used vertices. We denote by γ_2 the Hamiltonian path for \mathcal{Q}_n^{10} minus all deleted and used vertices that is defined by γ' and connects v_{10} to u_{10} .

There are $q_{11} + p - 1 + t + r - 1 = q_{11} + s - 2$ deleted or used even and odd vertices in \mathcal{Q}_n^{11} . Since $q_{11} + s - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from (T) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{11} minus all deleted and used vertices that connects v_{11} to u_{11} .

Let v'_{00} be any unused and undeleted odd vertex in \mathcal{Q}_n^{00} different from v_{00} which even neighbor u'_{01} in \mathcal{Q}_n^{01} is neither deleted nor used vertex. There are $q_{00} + s$ used or deleted even and $q_{00} + p - 1 + t + r - 1 + 1 = q_{00} + s - 1$ used or deleted odd vertices in \mathcal{Q}_n^{00} . Since $q_{00} + s - 1 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian

path γ_3 for \mathcal{Q}_n^{00} minus all deleted and used vertices that connects v_{00} to v'_{00} .

There are $q_{01} + p - 1 + t + r - 1 = q_{01} + s - 2$ used or deleted even and $q_{01} + p + t + r - 1 = q_{01} + s - 1$ deleted or used odd vertices in \mathcal{Q}_n^{01} (u_{01} and u'_{01} are not counted). Since $q_{01} + s - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted and used vertices that connects u'_{01} to u_{01} .

Finally, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \{u_1, \dots, u_k, v_1, \dots, v_k\}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} u_{11} \rightarrow v_{10} \xrightarrow{\gamma_2} u_{10} \rightarrow v_{00} \xrightarrow{\gamma_3} v'_{00} \rightarrow u'_{01} \xrightarrow{\gamma_4} u_{01}.$$

(d) \mathcal{Q}_n^{01} has a positive charge and \mathcal{Q}_n^{10} has a negative charge.

Let the charge of \mathcal{Q}_n^{01} be $p > 0$ and the charge of \mathcal{Q}_n^{10} be $r > 0$. Since there are odd vertices in at least three of the four plates either in \mathcal{Q}_n^{00} or in \mathcal{Q}_n^{01} there is a deleted odd vertex. Using the symmetry of this case, without loss of generality, we can assume that there is an odd deleted vertex in \mathcal{Q}_n^{00} . Then $q_{01} + p + q_{00} + s \leq k - 1$, hence $q_{01} + p + s \leq k - 2$ and therefore $p + s = t + r \leq k - 2$. Notice also that $q_{00} + p + s \leq k - 1$, $q_{10} + r + t \leq k - 1$ and $q_{11} + r + t \leq k - 1$, since there are even (odd) deleted vertices in at least three of the four plates.

Take a model path μ in \mathcal{Q}_n^{11} that begins with an even vertex $e_{11} = u_{11}$ and following μ make a total of $t + r - 2 = p + s - 2$ short cycles of the types

$$\begin{aligned} e_{11} &\rightarrow (o, e)_{01} \rightarrow o_{00} \rightarrow e_{10} \rightarrow (o, e')_{11}; \\ e_{11} &\rightarrow o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow (o, e')_{11}; \\ e_{11} &\rightarrow o_{01} \rightarrow (e, o)_{00} \rightarrow (e, o)_{10} \rightarrow e'_{11}; \\ e_{11} &\rightarrow (o, e)_{01} \rightarrow o_{00} \rightarrow (e, o)_{10} \rightarrow e'_{11}; \end{aligned}$$

such that at the end all plates to have charge one. We denote the end vertex of the resulting path by a_{11} .

Finally, begin with $e_{11} = a_{11}$ and following μ make the following path with length four

$$e_{11} \rightarrow (o, e)_{01} \rightarrow o_{00} \rightarrow e_{10}.$$

We denote the end vertex of the resulting path by u_{10} .

The total number of the constructed short cycles is $p - 1 + s - 1 + 1 = p + s - 1 \leq k - 3$, hence the length of μ is enough for that construction.

There are $q_{00} + p - 1 + s = q_{00} + p + s - 1$ deleted or used even and $q_{00} + p - 1 + s - 1 + 1 = q_{00} + p + s - 1$ deleted or used odd vertices in \mathcal{Q}_n^{00} . Since $q_{00} + p + s - 1 \leq k - 2 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{00} minus all deleted or used vertices. Let (u_{00}, v_{00}) be any edge in γ' such that the even neighbor u'_{10} of v_{00} in \mathcal{Q}_n^{10} and the odd neighbor v_{01} of u_{00} in \mathcal{Q}_n^{01} are neither deleted nor used vertices. We denote by γ_2 the Hamiltonian path for \mathcal{Q}_n^{00} minus all deleted and used vertices that is defined by γ' and connects v_{00} to u_{00} .

There are $q_{10} + r - 1 + t - 1 = q_{10} + r + t - 2$ used or deleted even and $q_{10} + r + t - 1$ deleted or used odd vertices in \mathcal{Q}_n^{10} (u_{10} and u'_{10} are not counted). Since $q_{10} + r + t - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{10} minus all deleted and used vertices that connects u_{10} to u'_{10} .

Let v'_{01} be any unused and undeleted odd vertex in \mathcal{Q}_n^{01} different from v_{01} such that its even neighbor u'_{11} in \mathcal{Q}_n^{11} is neither a deleted nor used vertex. There are $q_{01} + p + s - 1 + 1 = q_{01} + p + s$ deleted or used even and $q_{01} + p - 1 + s - 1 + 1 = q_{01} + p + s - 1$ deleted or used odd vertices in \mathcal{Q}_n^{01} . Since $q_{01} + p + s - 1 \leq (k - 2) - 1 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{01} minus all deleted and used vertices that connects v_{01} to v'_{01} .

There are $q_{11} + t - 1 + r - 1 = q_{11} + t + r - 2$ deleted or used even and $q_{11} + t + r - 1$ deleted or used odd vertices in \mathcal{Q}_n^{11} . Since $q_{11} + t + r - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{11} minus all deleted or used vertices that connects u'_{11} to u_{11} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{10} \xrightarrow{\gamma_1} u'_{10} \rightarrow v_{00} \xrightarrow{\gamma_2} u_{00} \rightarrow v_{01} \xrightarrow{\gamma_3} v'_{01} \rightarrow u'_{11} \xrightarrow{\gamma_4} u_{11}.$$

(2) K_1 and K_2 are at distance one.

Without loss of generality we can assume that $K_1 = \mathcal{Q}_n^{00}$ and $K_2 = \mathcal{Q}_n^{01}$. Then, up to symmetry and up to interchanging positive and negative charge, there are six different cases: (a) \mathcal{Q}_n^{10} and \mathcal{Q}_n^{11} are neutral; (b) \mathcal{Q}_n^{10} is neutral and \mathcal{Q}_n^{11} has a negative charge; (c) \mathcal{Q}_n^{10} has a negative charge and \mathcal{Q}_n^{11} is neutral; (d) \mathcal{Q}_n^{10} and \mathcal{Q}_n^{11} have negative charges; (e) \mathcal{Q}_n^{10} has a negative charge and \mathcal{Q}_n^{11} has a positive charge; and (f) \mathcal{Q}_n^{10} has a positive charge and \mathcal{Q}_n^{11} has a negative charge.

(a) \mathcal{Q}_n^{10} and \mathcal{Q}_n^{11} are neutral.

According to Assumption we do not need to consider this case.

(b) \mathcal{Q}_n^{10} is neutral and \mathcal{Q}_n^{11} has a negative charge.

This case is equivalent to case (1)(b).

(c) \mathcal{Q}_n^{10} has a negative charge and \mathcal{Q}_n^{11} is neutral.

Let the charge of \mathcal{Q}_n^{10} be $p > 0$, hence $p+t = s \leq k-2$ and $s+q_{11} \leq k-1$. Notice also that $q_{00} + s \leq k-2$, $q_{10} + p + t \leq k-1$ and $q_{01} + p + t \leq k-1$ since there are even (odd) deleted vertices in at least three of the four plates.

Take a model path μ in \mathcal{Q}_n^{01} that begins with an even vertex $e_{01} = u_{01}$ and following μ make $t-1$ short cycles of the type

$$e_{01} \rightarrow o_{00} \rightarrow (e, o)_{10} \rightarrow (e, o)_{11} \rightarrow e'_{01}.$$

We denote the end vertex of the resulting path by a_{01} .

Then begin with $e_{01} = a_{01}$ and following μ make $p-1$ short cycles of the type

$$e_{01} \rightarrow o_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow (o, e')_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Finally, begin with $e_{01} = a'_{01}$ and following μ make the following path with length three

$$e_{01} \rightarrow o_{00} \rightarrow e_{10} \rightarrow o_{11}.$$

We denote the end vertex of the resulting path by v_{11} .

The total number of the constructed short cycles is $t-1 + p-1 = t+p-2 \leq k-4$, hence the length of μ is enough for that construction.

There are $q_{10} + p - 1 + t - 1 + 1 = q_{10} + p + t - 1 \leq k - 2$ even and odd deleted or used vertices in \mathcal{Q}_n^{10} . Since $k-2 \leq (n-2)-2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{10} minus all deleted or used vertices. Let (u_{10}, v_{10}) be any edge in γ' such that the even neighbor u_{11} of v_{10} in \mathcal{Q}_n^{11} and the odd neighbor v_{00} of u_{10} in \mathcal{Q}_n^{00} are neither deleted nor used vertices. We denote by γ_2 the Hamiltonian path for \mathcal{Q}_n^{10} minus all deleted and used vertices that is defined by γ' and connects v_{10} to u_{10} .

There are $q_{11} + p - 1 + t - 1 = q_{11} + p + t - 2$ deleted or used even and odd vertices in \mathcal{Q}_n^{11} . Since $q_{11} + p + t - 2 \leq (k-1) - 2 \leq (n-2) - 3$, it follows from (T) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{11} minus all deleted and used vertices that connects v_{11} to u_{11} .

Let v'_{00} be any unused and undeleted odd vertex in \mathcal{Q}_n^{00} different from v_{00} which even neighbor u'_{01} in \mathcal{Q}_n^{01} is neither deleted nor used vertex.

There are $q_{00} + s$ used or deleted even and $q_{00} + p - 1 + t - 1 + 1 = q_{00} + s - 1$ used or deleted odd vertices in \mathcal{Q}_n^{00} . Since $q_{00} + s - 1 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{00} minus all deleted and used vertices that connects v_{00} to v'_{00} .

There are $q_{01} + p - 1 + t - 1 = q_{01} + p + t - 2$ used or deleted even and $q_{01} + p - 1 + t - 1 + 1 = q_{01} + p + t - 1$ deleted or used odd vertices in \mathcal{Q}_n^{01} (u_{01} and u'_{01} are not counted). Since $q_{01} + p + t - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted and used vertices that connects u'_{01} to u_{01} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} u_{11} \rightarrow v_{10} \xrightarrow{\gamma_2} u_{10} \rightarrow v_{00} \xrightarrow{\gamma_3} v'_{00} \rightarrow u'_{01} \xrightarrow{\gamma_4} u_{01}.$$

(d) \mathcal{Q}_n^{10} and \mathcal{Q}_n^{11} have negative charges.

This case is equivalent to case (1)(c).

(e) \mathcal{Q}_n^{10} has a negative charge and \mathcal{Q}_n^{11} has a positive charge.

Let the charge of \mathcal{Q}_n^{10} be $p > 0$ and the charge of \mathcal{Q}_n^{11} be $r > 0$. Since there are odd vertices in at least three of the four plates either in \mathcal{Q}_n^{00} or in \mathcal{Q}_n^{11} there is a deleted odd vertex. Using the symmetry of this case, without loss of generality, we can assume that there is an odd deleted vertex in \mathcal{Q}_n^{00} . Then $q_{00} + s + q_{11} + r \leq k - 1$, hence $q_{11} + s + r \leq k - 2$ and therefore $r + s = t + p \leq k - 2$. Notice also that $q_{00} + s + r \leq k - 1$, $q_{10} + t + p \leq k - 1$ and $q_{01} + t + p \leq k - 1$, since there are even (odd) deleted vertices in at least three of the four plates.

Take a model path μ in \mathcal{Q}_n^{01} that begins with an even vertex $e_{01} = u_{01}$ and following μ make a total of $t + p - 2 = r + s - 2$ short cycles of the types

$$\begin{aligned} e_{01} &\rightarrow (o, e)_{00} \rightarrow (o, e)_{10} \rightarrow o_{11} \rightarrow e'_{01}; \\ e_{01} &\rightarrow o_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow (o, e')_{01}; \\ e_{01} &\rightarrow o_{00} \rightarrow (e, o)_{10} \rightarrow (e, o)_{11} \rightarrow e'_{01}; \\ (e, o)_{01} &\rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow o_{11} \rightarrow e'_{01}; \end{aligned}$$

such that at the end all plates to have charge one. We denote the end vertex of the resulting path by a_{01} and let its odd neighbor in \mathcal{Q}_n^{00} be v_{00} . We extend the constructed path so far with the edge (a_{01}, v_{00}) .

The total number of the constructed short cycles is $r - 1 + s - 1 = r + s - 2 \leq k - 4$, hence the length of μ is enough for that construction.

Let v'_{00} be any unused and undeleted odd vertex in \mathcal{Q}_n^{00} different from v_{00} which even neighbor u_{10} in \mathcal{Q}_n^{10} is neither a deleted nor used vertex. There are $q_{00} + s + r - 1$ deleted or used even and $q_{00} + s + r - 2$ deleted or used odd vertices in \mathcal{Q}_n^{00} . Since $q_{00} + s + r - 2 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{00} minus all deleted and used vertices that connects v_{00} to v'_{00} .

Let u'_{10} be any unused and undeleted odd vertex in \mathcal{Q}_n^{10} different from u_{10} which odd neighbor v_{11} in \mathcal{Q}_n^{11} is neither a deleted nor used vertex. There are $q_{10} + p + t - 2$ deleted or used even and $q_{10} + p + t - 1$ deleted or used odd vertices in \mathcal{Q}_n^{10} . Since $q_{10} + p + t - 2 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_2 for \mathcal{Q}_n^{10} minus all deleted and used vertices that connects u_{10} to u'_{10} .

Let v'_{11} be any unused and undeleted odd vertex in \mathcal{Q}_n^{11} different from v_{11} which even neighbor u'_{01} in \mathcal{Q}_n^{01} is neither a deleted nor used vertex. There are $q_{11} + s + r - 1$ deleted or used even and $q_{11} + s + r - 2$ deleted or used odd vertices in \mathcal{Q}_n^{11} . Since $q_{11} + s + r - 2 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{11} minus all deleted and used vertices that connects v_{11} to v'_{11} .

There are $q_{01} + p + t - 2$ deleted or used even and $q_{01} + p + t - 1$ deleted or used odd vertices in \mathcal{Q}_n^{01} . Since $q_{01} + p + t - 2 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted and used vertices that connects u'_{01} to u_{01} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{00} \xrightarrow{\gamma_1} v'_{00} \rightarrow u_{10} \xrightarrow{\gamma_2} u'_{10} \rightarrow v_{11} \xrightarrow{\gamma_3} v'_{11} \rightarrow u'_{01} \xrightarrow{\gamma_4} u_{01}.$$

(f) \mathcal{Q}_n^{10} has a positive charge and \mathcal{Q}_n^{11} has a negative charge.

This case is equivalent to case (1)(d).

Note 2. For the remaining cases Remark 3.2 applies.

Case (E) There exists a column A in M_2 that separates the deleted even vertices in the way $(r, k - r)$, where $2 \leq r \leq k - 2$, and another column B in M_2 that separates the deleted odd vertices in different way than A and the deleted even vertices in the same way as A .

We split the hypercube using A and B . Without loss of generality, we can assume that the deleted vertices are distributed as follows:

$$\{v_1, \dots, v_p\} \subset \mathcal{Q}_n^{00}, \{u_1, \dots, u_r, v_{p+1}, \dots, v_s\} \subset \mathcal{Q}_n^{10}, \\ \{v_{s+1}, \dots, v_t\} \subset \mathcal{Q}_n^{11}, \text{ and } \{v_{t+1}, \dots, v_k, u_{r+1}, \dots, u_k\} \subset \mathcal{Q}_n^{01},$$

where $2 \leq r \leq k - 2$, $0 \leq p \leq k - 2$, $1 \leq s \leq k - 1$, and $2 \leq t \leq k$, since there are odd vertices in at least three of the hypercubes.

If $r \leq s - p < t$ then necessarily $k - r > k - t$. If $r > s - p$ then there are two possibilities: $k - r \leq k - t$ or $k - r > k - t$. Since the cases $r \leq s - p$, $k - r > k - t$ and $r > s - p$, $k - r \leq k - t$ are symmetric there are only two cases to consider: (1) $2 \leq r \leq s - p \leq k - 2$; and (2) $s - p < r \leq k - 2$ and $0 \leq k - t < k - r \leq k - 2$.

(1) $2 \leq r \leq s - p \leq k - 2$.

Take a model path μ in \mathcal{Q}_n^{01} that begins with an odd vertex $o_{01} = v_{01}$.

We know that there are odd vertices in at least three of the hypercubes and since $s - p \geq 2$ we conclude that there are odd vertices in \mathcal{Q}_n^{10} . Hence there are three possibilities: (a) $t - s \geq 1$ and $k - t \geq 1$; (b) $t - s \geq 1$ and $p \geq 1$; and (c) $k - t \geq 1$ and $p \geq 1$. Since case (c) is symmetric to case (a) we consider only cases (a) and (b).

To construct a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ in each one of these cases we proceed as follows.

(a) $t - s \geq 1$ and $k - t \geq 1$.

In this case $3 \leq s + 1 \leq t \leq k - 1$.

Begin with $o_{01} = v_{01}$ and following μ make $s - p - r$ short cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Now begin with $o_{01} = a'_{01}$ and following μ make p cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a''_{01} .

Finally, begin with $o_{01} = a''_{01}$ and following μ make $t - s - 1$ cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow (e, o)_{10} \rightarrow e_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'''_{01} .

Let the neighbor of a'''_{01} in \mathcal{Q}_n^{00} be u_{00} . We extend the constructed path with the edge (a'''_{01}, u_{00}) .

The total number of the constructed short cycles is $s - p - r + p + t - s - 1 = t - r - 1 \leq k - 3$, hence the length of μ is enough for that construction.

There are $s - p - r + p + r + t - s - 1 = t - 1$ used or deleted even and $s - p + p + t - s - 1 = t - 1$ odd used or deleted vertices in \mathcal{Q}_n^{10} . Since $t - 1 \leq (k - 1) - 1 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{10} minus all deleted or used vertices. Let (u_{10}, v_{10}) be any edge in γ' such that the even neighbor u_{11} of v_{10} in \mathcal{Q}_n^{11} and the odd neighbor v_{00} of u_{10} in \mathcal{Q}_n^{00} are neither deleted nor used vertices. Such edge exists since there are only p deleted odd vertices in \mathcal{Q}_n^{00} that could be neighbors of u_{10} and should be avoided. We denote by γ_2 the Hamiltonian path for \mathcal{Q}_n^{10} minus all deleted or used vertices that is defined by γ' and connects u_{10} to v_{10} .

There are $s - p - r + p + t - s - 1 = t - r - 1$ even and odd deleted or used vertices in \mathcal{Q}_n^{00} (u_{00} is not counted). Since $t - r - 1 \leq (k - 1) - 2 - 1 \leq (n - 2) - 4$, it follows from (T) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{00} minus all deleted or used vertices that connects u_{00} to v_{00} .

There are $s - p - r + p + t - s - 1 = t - r - 1$ used even and $s - p - r + p + t - s = t - r$ odd deleted or used vertices in \mathcal{Q}_n^{11} . Let u'_{11} be any unused even vertex in \mathcal{Q}_n^{11} such that its odd neighbor v'_{01} in \mathcal{Q}_n^{01} is neither a deleted nor used vertex. Since $t - r - 1 \leq (k - 1) - 2 - 1 \leq (n - 2) - 4$, it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{11} minus all deleted or used vertices that connects u_{11} to u'_{11} .

There are $k - r$ deleted even and $s - p - r + p + t - s - 1 + k - t = k - r - 1$ deleted or used odd vertices in \mathcal{Q}_n^{01} . Since $k - r - 1 \leq k - 2 - 1 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted or used vertices that connects v'_{01} to v_{01} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{00} \xrightarrow{\gamma_1} v_{00} \rightarrow u_{10} \xrightarrow{\gamma_2} v_{10} \rightarrow u_{11} \xrightarrow{\gamma_3} u'_{11} \rightarrow v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

(b) $t - s \geq 1$ and $p \geq 1$.

In this case $t = k$.

Begin with $o_{01} = v_{01}$ and following μ make $s - p - r$ short cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Now begin with $o_{01} = a'_{01}$ and following μ make $p - 1$ cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a''_{01} .

Finally, begin with $o_{01} = a''_{01}$ and following μ make $t - s - 1$ cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow (e, o)_{10} \rightarrow e_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'''_{01} . Following μ we extend the constructed path with the path

$$a'''_{01} \rightarrow (e, o)_{00} \rightarrow e_{10}.$$

We denote the end vertex of the resulting path by u_{10} .

The total number of the constructed short cycles is $s - p - r + p - 1 + t - s - 1 = t - r - 2 \leq k - 4$, hence the length of μ is enough for that construction.

There are $s - p - r + p - 1 + r + t - s - 1 = t - 2$ used or deleted even and $s - p + p - 1 + t - s - 1 = t - 2$ odd used or deleted vertices in \mathcal{Q}_n^{10} . Since $t - 2 \leq k - 2 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{10} minus all deleted or used vertices. This cycle contains the vertex u_{10} . Let v_{10} be a vertex which is a neighbor of u_{10} in γ' . Clearly the even neighbor u_{11} of v_{10} in \mathcal{Q}_n^{11} is neither a deleted nor used vertex. We denote by γ_1 the Hamiltonian path for \mathcal{Q}_n^{10} minus all deleted or used vertices that is defined by γ' and connects u_{10} to v_{10} .

There are $s - p - r + p - 1 + t - s - 1 = t - r - 2$ used even and $s - p - r + p - 1 + t - s = t - r - 1$ odd deleted or used vertices in \mathcal{Q}_n^{11} . Let u'_{11} be any unused even vertex in \mathcal{Q}_n^{11} such that its odd neighbor v'_{01} in \mathcal{Q}_n^{01} is neither a deleted nor used vertex. Clearly, the even neighbor u_{00} of v'_{01} in \mathcal{Q}_n^{00} is also neither a deleted nor used vertex. Since $t - r - 2 \leq k - 2 - 2 \leq (n - 2) - 4$, it follows from (CG) that there exists a Hamiltonian path γ_2 for \mathcal{Q}_n^{11} minus all deleted or used vertices that connects u_{11} to u'_{11} .

There are $s - p - r + p - 1 + t - s - 1 = t - r - 2$ even and $s - p - r + t - s - 1 + p = t - r - 1$ odd deleted or used vertices in \mathcal{Q}_n^{00} (u_{00} is not counted). Let u'_{00} be any unused even vertex in \mathcal{Q}_n^{00} such that its odd neighbor v''_{01} in \mathcal{Q}_n^{01} is neither a deleted nor used vertex. Since $t - r - 2 \leq k - 2 - 2 \leq (n - 2) - 4$, it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{00} minus all deleted or used vertices that connects u_{00} to u'_{00} .

There are $k - r$ deleted even and $s - p - r + p - 1 + t - s - 1 + 1 + k - t = k - r - 1$ deleted or used odd vertices in \mathcal{Q}_n^{01} . Since $k - r - 1 \leq k - 2 - 1 \leq (n - 2) - 3$, it follows from (CG) that there

exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted or used vertices that connects v_{01}'' to v_{01} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{10} \xrightarrow{\gamma_1} v_{10} \rightarrow u_{11} \xrightarrow{\gamma_2} u'_{11} \rightarrow v'_{01} \rightarrow u_{00} \xrightarrow{\gamma_3} u'_{00} \rightarrow v_{01}'' \xrightarrow{\gamma_4} v_{01}.$$

(2) $s - p < r \leq k - 2$ and $0 \leq k - t < k - r \leq k - 2$.

Since there are odd vertices in at least three of the hypercubes and because of the symmetry, without loss of generality, we can assume that $p \geq 1$. Again thanks to the symmetrical situation we can also assume that $p \geq t - s$ and $r - (s - p) \leq k - r - (k - t)$. Since $p + (t - s) = (r - (s - p)) + (k - r - (k - t))$ then either $t - s \leq r - (s - p) \leq k - r - (k - t) \leq p$ or $r - (s - p) \leq t - s \leq p \leq k - r - (k - t)$. Since both cases are symmetrical we consider only the case

$$t - s \leq r - (s - p) \leq k - r - (k - t) \leq p.$$

There are two possibilities: (a) $p \geq t - s \geq 1$; and (b) $p \geq 2$ and $t - s = 0$.

(a) $p \geq t - s \geq 1$.

Either $t < k$ or $s - p > 0$. Since both cases are symmetrical we consider below only the case $s - p > 0$, hence $k - (s - p) - 2 \leq k - 3$.

Begin with $o_{01} = v_{01}$ and following μ make $t - r - 1$ short cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Now begin with $o_{01} = a'_{01}$ and following μ make $t - s - 1$ cycles of the type

$$(o, e)_{01} \rightarrow (o, e)_{00} \rightarrow o_{10} \rightarrow e_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a''_{01} .

Finally, begin with $o_{01} = a''_{01}$ and following μ make $p - t + r$ cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow o_{10} \rightarrow (e, o)_{11} \rightarrow (e, o')_{01}.$$

We denote the end vertex of the resulting path by a'''_{01} . Let the neighbor of a'''_{01} in \mathcal{Q}_n^{00} be u_{00} . We extend the constructed path with the edge (a'''_{01}, u_{00}) .

The total number of the constructed short cycles is $t - r - 1 + t - s - 1 + p - t + r = t - (s - p) - 2$. Since there are odd vertices in at least

three hypercubes either $t < k$ or $s > p$, hence $t - (s - p) - 2 \leq k - 3$ and therefore the length of μ is enough for that construction.

There are $t - r - 1 + r = t - 1$ even and $t - r - 1 + t - s - 1 + p - t + r + s - p = t - 2$ used or deleted odd vertices in \mathcal{Q}_n^{10} . Also, there are $k - r + t - s - 1 + p - t + r = k - (s - p) - 1$ deleted or used even and $k - t + t - r - 1 + t - s - 1 + p - t + r = k - (s - p) - 2$ deleted or used odd vertices in \mathcal{Q}_n^{01} .

We fix one deleted vertex u in \mathcal{Q}_n^{10} . Since $t - 2 \leq k - 2 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{10} minus all deleted or used vertices except u . This cycle contains u . Let v_{10} and v'_{10} be the neighbors of u in γ' . Clearly the even neighbor u_{11} of v'_{10} in \mathcal{Q}_n^{11} and u'_{00} of v_{10} in \mathcal{Q}_n^{00} are neither deleted nor used vertices. We denote by γ_2 the Hamiltonian path for \mathcal{Q}_n^{10} minus all deleted or used vertices that is defined by γ' and connects v_{10} to v'_{10} . Let u'_{11} be any unused even vertex in \mathcal{Q}_n^{11} such that its odd neighbor v'_{01} in \mathcal{Q}_n^{01} is neither deleted nor used vertex. Since $k - (s - p) - 2 \leq k - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted or used vertices that connects v'_{01} to v_{01} .

There are $t - r - 1 + t - s - 1 + p - t + r = t - (s - p) - 2$ even and $t - s - 1 + p = t - (s - p) - 1$ odd deleted or used vertices in \mathcal{Q}_n^{00} (u_{00} is not counted). Since $t - (s - p) - 2 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{00} minus all deleted or used vertices that connects u_{00} to u'_{00} .

There are $t - r - 1 + t - s - 1 + p - t + r = t - (s - p) - 2$ used even and $t - r - 1 + p - t + r + t - s = t - (s - p) - 1$ odd deleted or used vertices in \mathcal{Q}_n^{11} . Since $t - (s - p) - 2 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{11} minus all deleted or used vertices that connects u_{11} to u'_{11} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{00} \xrightarrow{\gamma_1} u'_{00} \rightarrow v_{10} \xrightarrow{\gamma_2} v'_{10} \rightarrow u_{11} \xrightarrow{\gamma_3} u'_{11} \rightarrow v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

(b) $p \geq 2$ and $t - s = 0$.

Then $t < k$ and $s - p > 0$, hence $t - 1 \leq k - 2$ and $k - (s - p) - 2 \leq k - 3$.

Begin with $o_{01} = v_{01}$ and following μ make $t - r - 1$ short cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Finally, begin with $o_{01} = a'_{01}$ and following μ make $r - (s - p) - 1$ cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow o_{10} \rightarrow (e, o)_{11} \rightarrow (e, o')_{01}.$$

We denote the end vertex of the resulting path by a''_{01} . Let the neighbor of a''_{01} in \mathcal{Q}_n^{00} be e_{00} , the neighbor of e_{00} in \mathcal{Q}_n^{10} be o_{10} , and the neighbor of o_{10} in \mathcal{Q}_n^{11} be u_{11} . We extend the constructed path with the path

$$v''_{01} \rightarrow e_{00} \rightarrow o_{10} \rightarrow u_{11}.$$

The total number of the constructed short cycles is $t - r - 1 + r - (s - p) - 1 = t - s + p - 2 = p - 2 \leq (k - 2) - 2 = k - 4$, hence the length of μ is enough for that construction.

There are $t - r - 1 + r = t - 1$ used or deleted even and $t - r - 1 + r - s + p - 1 + s - p + 1 = t - 1$ used or deleted odd vertices in \mathcal{Q}_n^{10} . Since $t - 1 \leq k - 2 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{10} minus all deleted or used vertices. Let u_{10} and v_{10} be two neighbors in γ' . Clearly the even neighbor v_{11} of u_{10} in \mathcal{Q}_n^{11} and u_{00} of v_{10} in \mathcal{Q}_n^{00} are neither deleted nor used vertices. We denote by γ_2 the Hamiltonian path for \mathcal{Q}_n^{10} minus all deleted or used vertices that is defined by γ' and connects u_{10} to v_{10} .

There are $t - r - 1 + r - s + p - 1 = t - s + p - 2 = p - 2$ used even and odd deleted or used vertices in \mathcal{Q}_n^{11} . Since $p - 2 \leq k - 4 \leq (n - 2) - 4$, it follows from (T) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{11} minus all deleted or used vertices that connects u_{11} to v_{11} .

There are $t - r - 1 + r - s + p - 1 = t - s + p - 2 = p - 2$ even and $t - s - 1 + p = t - s + p - 1 = p - 1$ odd deleted or used vertices in \mathcal{Q}_n^{00} (u_{00} is not counted). Let u'_{00} be any unused even vertex in \mathcal{Q}_n^{00} such that its odd neighbor v'_{01} in \mathcal{Q}_n^{01} is neither deleted nor used vertex. Since $p - 2 \leq k - 2 - 2 \leq (n - 2) - 4$, it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{00} minus all deleted or used vertices that connects u_{00} to u'_{00} .

Also, there are $k - r + r - s + p - 1 = k - (s - p) - 1$ deleted or used even and $k - t + t - r - 1 + r - s + p - 1 = k - (s - p) - 2$ deleted or used odd vertices in \mathcal{Q}_n^{01} . Since $k - (s - p) - 2 \leq k - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted or used vertices that connects v'_{01} to v_{01} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{11} \xrightarrow{\gamma_1} v_{11} \rightarrow u_{10} \xrightarrow{\gamma_2} v_{10} \rightarrow u_{00} \xrightarrow{\gamma_3} u'_{00} \rightarrow v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

Case (F) There exists a column A in M_2 that separates the even vertices in the way $(r, k - r)$, where $2 \leq r \leq k - 2$, and a column B in M_o that separates the odd vertices in a different way than A . Also, either A or B separates the odd vertices in the way $(s, k - s)$, where $2 \leq s \leq k - 2$.

If B separates the deleted odd vertices in the way $(1, k - 1)$ and all deleted vertices in the way $(1, 2k - 1)$ then that would be case (B) . If B separates the deleted odd vertices in the way $(2, k - 2)$ and all deleted vertices in the way $(2, 2k - 2)$ then that would be case (C) . Therefore we assume that B separates the deleted odd vertices in the way $(s, k - s)$ and all deleted vertices in the way $(s, 2k - s)$, where $3 \leq s \leq k - 1$.

We split the hypercube using A and B . Without loss of generality, we can assume that the deleted vertices are distributed as follows:

$$\begin{aligned} \{v_1, \dots, v_p\} &\subset \mathcal{Q}_n^{00}, \{v_{t+1}, \dots, v_k, u_{r+1}, \dots, u_k\} \subset \mathcal{Q}_n^{01}, \\ \{v_{p+1}, \dots, v_s\} &\subset \mathcal{Q}_n^{10}, \text{ and } \{u_1, \dots, u_r, v_{s+1}, \dots, v_t\} \subset \mathcal{Q}_n^{11}, \end{aligned}$$

where $2 \leq r \leq k - 2$, $0 \leq p \leq k - 2$, $3 \leq s \leq k - 1$, and $2 \leq t \leq k$, since there are odd vertices in at least three of the hypercubes. Also, either $k - r > k - t$ or $r > t - s$ since in at least one of the hypercubes \mathcal{Q}_n^{00} or \mathcal{Q}_n^{10} there is an odd vertex. Without loss of generality, we assume that $k - r > k - t$. Finally, it follows from our assumptions how A and B separate the odd vertices that either $k - s \geq 2$ or $2 \leq t - p \leq k - 2$.

We consider two cases.

(1) $r \leq t - s$.

There are two sub cases to consider.

(a) $t \leq k - 1$, hence there is at least one odd vertex in \mathcal{Q}_n^{01} .

(i) There is an odd vertex in \mathcal{Q}_n^{00} , hence $p \geq 1$.

Take a model path μ in \mathcal{Q}_n^{01} that begins with an odd vertex $o_{01} = v_{01}$ and following μ make $p - 1$ short cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Now begin with $o_{01} = a'_{01}$ and following μ make $s - p$ cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a''_{01} .

Finally, begin with $o_{01} = a''_{01}$ and following μ make $t - s - r$ cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow (e, o)_{10} \rightarrow e_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'''_{01} . Let the neighbor of a'''_{01} in \mathcal{Q}_n^{00} be u_{00} . We extend the constructed path with the edge (a'''_{01}, u_{00}) .

The total number of the constructed short cycles is $p - 1 + s - p + t - s - r = t - r - 1 \leq k - 3$, hence the length of μ is enough for that construction.

There are $p - 1 + s - p + t - s - r + r = t - 1$ deleted or used even and $p - 1 + s - p + t - s = t - 1$ used or deleted odd vertices in \mathcal{Q}_n^{11} . Since $t - 1 \leq k - 1 - 1 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{11} minus all deleted or used vertices. Let u_{11} and v_{11} be two neighbors in γ' such that the odd neighbor v'_{01} of u_{11} in \mathcal{Q}_n^{01} is different from v_{01} and is neither deleted nor used vertex. Clearly the even neighbor u_{10} of v_{11} in \mathcal{Q}_n^{10} is also neither deleted nor used vertex. We denote by γ_3 the Hamiltonian path for \mathcal{Q}_n^{11} minus all deleted or used vertices that is defined by γ' and connects v_{11} to u_{11} .

Let u'_{00} be neither deleted nor used vertex in \mathcal{Q}_n^{00} different from u_{00} , which neighbor v_{10} in \mathcal{Q}_n^{10} is neither deleted nor used vertex. There are $p - 1 + s - p + t - s - r = t - r - 1$ used even and $p + s - p + t - s - r = t - r$ deleted or used odd vertices in \mathcal{Q}_n^{00} . Since $t - r - 1 \leq k - 1 - 3 \leq (n - 2) - 4$, it follows from (CG) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{00} minus all deleted or used vertices that connects u_{00} to u'_{00} .

There are $p - 1 + s - p + t - s - r = t - r - 1$ used even and $p - 1 + t - s - r + s - p = t - r - 1$ deleted or used odd vertices in \mathcal{Q}_n^{10} . Since $t - r - 1 \leq k - 1 - 3 \leq (n - 2) - 4$, it follows from (T) that there exists a Hamiltonian path γ_2 for \mathcal{Q}_n^{10} minus all deleted or used vertices that connects v_{10} to u_{10} .

There are $k - r$ deleted even and $p - 1 + s - p + t - s - r + k - t = k - r - 1$ odd deleted or used vertices in \mathcal{Q}_n^{01} (v_{01} is not counted). Since $k - r - 1 \leq k - 1 - 3 \leq (n - 2) - 4$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted or used vertices that connects v'_{01} to v_{01} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - F$ we extend the previously constructed path with the path

$$u_{00} \xrightarrow{\gamma_1} u'_{00} \rightarrow v_{10} \xrightarrow{\gamma_2} u_{10} \rightarrow v_{11} \xrightarrow{\gamma_3} u_{11} \rightarrow v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

(ii) There is an odd vertex in \mathcal{Q}_n^{10} , hence $s - p \geq 1$.

This case is similar to the previous case in (i). To obtain a solution of that case just switch the roles of \mathcal{Q}_n^{00} and \mathcal{Q}_n^{10} in the above solution. Then in the beginning of the construction make p cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01},$$

instead of $p - 1$ and then make $s - p - 1$ cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01},$$

instead of $s - p$ cycles. At the end finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ by extending the previously constructed path with the path

$$u_{00} \xrightarrow{\gamma_1} v_{00} \rightarrow u_{10} \xrightarrow{\gamma_2} u'_{10} \rightarrow v_{11} \xrightarrow{\gamma_3} u_{11} \rightarrow v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

(b) $t = k$, hence there are no odd vertices in \mathcal{Q}_n^{01} .

Since there are no odd vertices in \mathcal{Q}_n^{01} , there must be odd vertices in the other three hypercubes. Also, since $2 \leq r \leq k - 2$, the difference between deleted even and odd vertices in \mathcal{Q}_n^{01} is $k - r \geq 2$.

Take a model path μ in \mathcal{Q}_n^{00} that begins with an even vertex $e_{00} = u_{00}$ and following μ make $t - s - r$ short cycles of the type

$$e_{00} \rightarrow o_{01} \rightarrow e_{11} \rightarrow (o, e)_{10} \rightarrow (o, e')_{00}.$$

We denote the end vertex of the resulting path by b'_{00} .

Now begin with $e_{00} = b'_{00}$ and following μ make $s - p - 1$ cycles of the type

$$e_{00} \rightarrow o_{01} \rightarrow (e, o)_{11} \rightarrow e_{10} \rightarrow (o, e')_{00}.$$

We denote the end vertex of the resulting path by b''_{00} .

Finally, begin with $e_{00} = b''_{00}$ and following μ make $p - 1$ cycles of the type

$$e_{00} \rightarrow o_{01} \rightarrow (e, o)_{11} \rightarrow (e, o)_{10} \rightarrow e'_{00}.$$

We denote the end vertex of the resulting path by b'''_{00} . Let the neighbor of b'''_{00} in \mathcal{Q}_n^{01} be v''_{01} and the neighbor of v''_{01} in \mathcal{Q}_n^{11} be u_{11} . We extend the constructed path with the edges (b'''_{00}, v''_{01}) and (v''_{01}, u_{11}) .

The total number of the constructed short cycles is $p - 1 + s - p - 1 + t - s - r = t - r - 2 \leq k - 4$, hence the length of μ is enough for that construction.

There are $p - 1 + s - p - 1 + t - s - r + r = t - 2 = k - 2$ deleted or used even and $p - 1 + s - p - 1 + t - s = t - 2 = k - 2$ used or deleted odd vertices in \mathcal{Q}_n^{11} . Since $k - 2 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{11} minus all deleted

or used vertices. Clearly γ' contains u_{11} . Let v_{11} be a neighbor of u_{11} in γ' . Then the even neighbor u_{10} of v_{11} in \mathcal{Q}_n^{10} is neither deleted nor used vertex. We denote by γ_1 the Hamiltonian path for \mathcal{Q}_n^{11} minus all deleted or used vertices that is defined by γ' and connects u_{11} to v_{11} .

Let e''_{00} and o''_{00} be two neighbors in γ' such that the odd neighbor v_{01} of e''_{00} in \mathcal{Q}_n^{01} is neither deleted nor used vertex. Clearly the even neighbor u'_{10} of o''_{00} in \mathcal{Q}_n^{10} is also neither deleted nor used vertex. There are $p - 1 + s - p - 1 + t - s - r = t - r - 2$ used even and $p - 1 + t - s - r + s - p = t - r - 1$ deleted or used odd vertices in \mathcal{Q}_n^{10} . Since $t - r - 2 \leq k - 2 - 2 \leq (n - 2) - 4$, it follows from (CG) that there exists a Hamiltonian path γ_2 for \mathcal{Q}_n^{10} minus all deleted or used vertices that connects u_{10} to u'_{10} .

Let v'_{01} be neither deleted nor used vertex in \mathcal{Q}_n^{01} different from v_{01} , which neighbor u'_{00} in \mathcal{Q}_n^{00} is neither deleted nor used vertex. There are $k - r$ deleted even and $p - 1 + s - p - 1 + t - s - r + 1 + k - t = k - r - 1$ odd deleted or used vertices in \mathcal{Q}_n^{01} (v_{01} is not counted). Since $k - r - 1 \leq k - 2 - 1 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{01} minus all deleted or used vertices that connects v_{01} to v'_{01} .

There are $p - 1 + s - p - 1 + t - s - r + 1 = t - r - 1$ used even and $p + s - p - 1 + t - s - r + 1 = t - r$ deleted or used odd vertices in \mathcal{Q}_n^{00} . Since $t - r - 1 \leq k - 2 - 1 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{00} minus all deleted or used vertices that connects u'_{00} to u_{00} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{11} \xrightarrow{\gamma_1} v_{11} \rightarrow u_{10} \xrightarrow{\gamma_2} u'_{10} \rightarrow o''_{00} \rightarrow e''_{00} \rightarrow v_{01} \xrightarrow{\gamma_3} v'_{01} \rightarrow u'_{00} \xrightarrow{\gamma_4} u_{00}.$$

(2) $r > t - s$.

We have $k - r > k - t$ and $r > t - s$. Since $(t - r) + r - (t - s) = s \geq 3$ we have either $(k - r) - (k - t) = t - r \geq 2$ or $r - (t - s) \geq 2$. Also, since $s \geq 3$, we have either $p \geq 2$ or $s - p \geq 2$. There are two possibilities: either $(k - r) - (k - t) = t - r \geq 2$ and $p \geq 2$ (or equivalently, $r - (t - s) \geq 2$ and $s - p \geq 2$) or we do not have any of the previous cases and we have $(k - r) - (k - t) = t - r \geq 2$ and $p \leq 1$ (or $r - (t - s) \geq 2$ and $s - p \leq 1$) instead.

Since the cases in each group are symmetric of each other we consider only the first cases from each group.

(a) $(k - r) - (k - t) = t - r \geq 2$ and $p \geq 2$.

For easier explanation of how we balance the plates we assume that $(k - r) - (k - t) = t - r \geq r - (t - s)$. The other case is similar: the balancing of the plates is slightly different but the rest of the construction is the same.

There are three possibilities:

(i) $t - r \geq \max(p, s - p)$, (ii) $s - p \leq t - r \leq p$, and (iii) $p \leq t - r \leq s - p$.

We consider all cases below.

In case (i) we have $t - r \geq \max(p, s - p) \geq \min(p, s - p) \geq r - t + s$.

Take a model path μ in \mathcal{Q}_n^{01} that begins with an odd vertex $o_{01} = v_{01}$ and following μ make $r - t + s - 1$ short cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow o_{11} \rightarrow (e, o')_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Now begin with $o_{01} = a'_{01}$ and following μ make $t - p - r$ cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a''_{01} .

Then, begin with $o_{01} = a''_{01}$ and following μ make $p - 2$ cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'''_{01} .

Finally, begin with $o_{01} = a'''_{01}$ and following μ , extend the resulting path with the following path with length four

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow o_{11}.$$

We denote the end vertex of the resulting path by v_{11} .

In case (ii) we have

$$s - p \leq r - t + s \leq t - r \leq p.$$

In that case, begin with $o_{01} = v_{01}$ and following μ make $s - p - 1$ short cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow o_{11} \rightarrow (e, o')_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Now begin with $o_{01} = a'_{01}$ and following μ make $p + r - t$ cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow o_{11} \rightarrow (e, o')_{01}.$$

We denote the end vertex of the resulting path by a''_{01} .

Then, begin with $o_{01} = a''_{01}$ and following μ make $t - r - 2$ cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'''_{01} . Let the neighbor of a'''_{01} in \mathcal{Q}_n^{00} be u_{00} .

Finally, begin with $o_{01} = a'''_{01}$ and following μ , extend the resulting path with the following path with length four

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow o_{11}.$$

We denote the end vertex of the resulting path by v_{11} .

In case (iii) we have

$$p \leq r - t + s \leq t - r \leq s - p.$$

In that case, begin with $o_{01} = v_{01}$ and following μ make $p - 2$ short cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow o_{11} \rightarrow (e, o')_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Now begin with $o_{01} = a'_{01}$ and following μ make $r - t + s - p + 1$ cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow o_{11} \rightarrow (e, o')_{01}.$$

We denote the end vertex of the resulting path by a''_{01} .

Then, begin with $o_{01} = a''_{01}$ and following μ make $t - r - 2$ cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'''_{01} . Let the neighbor of a'''_{01} in \mathcal{Q}_n^{00} be u_{00} .

Finally, begin with $o_{01} = a'''_{01}$ and following μ , extend the resulting path with the following path with length four

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow o_{11}.$$

We denote the end vertex of the resulting path by v_{11} .

The total number of the constructed short cycles is $r - t + s - 1 + t - p - r + p - 2 = s - 3$ in (i), $s - p - 1 + p + r - t + t - r - 2 = s - 3$

in (ii), and $p - 2 + r - t + s - p + 1 + t - r - 2 = s - 3$ in (iii). Since $s - 3 \leq k - 4 \leq n - 6$, the length of μ is enough for that construction.

Let v'_{11} be neither deleted nor used odd vertex in \mathcal{Q}_n^{00} different from v_{11} . Clearly, its neighbor u_{10} in \mathcal{Q}_n^{10} is not an used vertex. There are $r - t + s + t - p - r + p - 2 = s - 2$ deleted or used even and $r - t + s - 1 + t - p - r + p - 2 = s - 3$ used or deleted odd vertices in \mathcal{Q}_n^{11} . Since $s - 3 \leq k - 1 - 3 \leq (n - 2) - 4$, it follows from (CG) that there exists a Hamiltonian path γ' for \mathcal{Q}_n^{11} minus all deleted or used vertices that connects v_{11} to v'_{11} . Let v''_{11} and u_{11} be two neighbors in γ' such that v''_{11} is closer to v'_{11} and the odd neighbor v'_{01} of u_{11} in \mathcal{Q}_n^{01} is different from v_{01} and is neither deleted nor used vertex. Clearly the even neighbor u'_{10} of v''_{11} in \mathcal{Q}_n^{10} is also neither deleted nor used vertex. We denote by γ_1 the path in \mathcal{Q}_n^{11} defined by γ' that connects v_{11} to v''_{11} and by γ_2 the path defined by γ' that connects v'_{11} to u_{11} .

There are $r - t + s - 1 + t - p - r + p - 2 + 1 = s - 2$ used even and $s - p + p - 2 + 1 = s - 1$ deleted or used odd vertices in \mathcal{Q}_n^{10} . Since $s - 2 \leq k - 1 - 2 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{10} minus all deleted or used vertices that connects u'_{10} to u_{10} .

Let u_{00} be the even neighbor of v'_{01} in \mathcal{Q}_n^{00} . Clearly, u_{00} is not a deleted vertex. Let u'_{00} be another unused even vertex in \mathcal{Q}_n^{10} different from u_{00} and such that its odd neighbor v''_{01} in \mathcal{Q}_n^{10} is neither deleted nor used vertex. There are $r - t + s - 1 + t - p - r + p - 2 + 1 = s - 2$ used even and $r - t + s - 1 + t - p - r + p = s - 1$ deleted or used odd vertices in \mathcal{Q}_n^{00} . Since $s - 2 \leq k - 1 - 2 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{00} minus all deleted or used vertices that connects u_{00} to u'_{00} .

There are $k - r + r - t + s - 1 = k - t + s - 1 \leq s - 1$ deleted even and $r - t + s - 1 + t - p - r + p - 2 + 1 = s - 2$ odd deleted or used vertices in \mathcal{Q}_n^{01} (v_{01} is not counted). Since $s - 2 \leq k - 1 - 2 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_5 for \mathcal{Q}_n^{01} minus all deleted or used vertices that connects v''_{01} to v_{01} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$\begin{aligned} v_{11} &\xrightarrow{\gamma_1} v''_{11} \rightarrow u'_{10} \xrightarrow{\gamma_3} u_{10} \rightarrow v'_{11} \xrightarrow{\gamma_2} u_{11} \rightarrow \\ &v'_{01} \rightarrow u_{00} \xrightarrow{\gamma_4} u'_{00} \rightarrow v''_{01} \xrightarrow{\gamma_5} v_{01}. \end{aligned}$$

(b) $(k - r) - (k - t) = t - r \geq 2$ and $p \leq 1$.

Since $s \geq 3$ and $p \leq 1$ we have $s - p \geq 2$. Therefore if $r - t + s \geq 2$ then that would be case (a). Thus, we have $r - t + s = 1 \geq p$ and since $r \geq 2$ we conclude that $t - s \geq 1$, hence there exists at least one odd deleted vertex in \mathcal{Q}_n^{11} . Also, it follows that $t - r > r - t + s$. Finally, since $s \geq 3$ we have $s - p \geq 2$, hence $p \leq s - p$. Therefore we have $p \leq r - t + s \leq t - r \leq s - p$.

There are two cases: $t - s \geq 2$ or $t - s = 1$. If $t - s \geq 2$ then $s \leq k - 2$. Let $t - s = 1$. Since there exists at most one deleted odd vertex in \mathcal{Q}_n^{00} , there must be at least one deleted odd vertex in \mathcal{Q}_n^{01} for at least one of both coordinates A or B separates the deleted vertices in two groups with at least two deleted odd vertices in each group. Hence, again $s \leq k - 2$. Therefore in either case we have $s \leq k - 2$.

We consider two cases below: (i) $t = k$ and therefore there are no deleted odd vertices in \mathcal{Q}_n^{01} , hence $p = 1$; and (ii) $t \leq k - 1$ and therefore there is at least one deleted odd vertex in \mathcal{Q}_n^{01} , hence $p \leq 1$.

(i) $t = k$ and therefore there are no deleted odd vertices in \mathcal{Q}_n^{01} , hence $p = 1$.

Take a model path μ in \mathcal{Q}_n^{01} that begins with an odd vertex $o_{01} = v_{01}$ and following μ make $t - r - 2 = k - r - 2$ short cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Then begin with $o_{01} = a'_{01}$ and following μ make $r - t + s = r + s - k$ cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow o_{11} \rightarrow (e, o')_{01}.$$

We denote the end vertex of the resulting path by a''_{01} .

Finally, begin with $o_{01} = a''_{01}$ and following μ , extend the resulting path with the following path with length four

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow o_{11}.$$

We denote the end vertex of the resulting path by v_{11} .

The total number of the constructed short cycles is $k - r - 2 + r + s - k + 1 = s - 1 \leq (k - 2) - 1 \leq k - 3$, hence the length of μ is enough for that construction.

There are $k - r - 2 + r = k - 2$ deleted or used even and $k - r - 2 + r + s - k + k - s = k - 2$ used or deleted odd vertices in \mathcal{Q}_n^{11} (v_{11} is not counted). Since $k - 2 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{11} minus all deleted or used vertices.

This cycle contains v_{11} . Let u_{11} be a neighbor of v_{11} in γ' . We denote by γ_1 the Hamiltonian path for \mathcal{Q}_n^{11} minus all deleted or used vertices defined by γ' that connects v_{11} to u_{11} .

Clearly the odd neighbor v'_{01} in \mathcal{Q}_n^{01} of u_{11} is not an used vertex and not a deleted vertex since there are no deleted odd vertices in \mathcal{Q}_n^{01} . Let u_{00} be the even neighbor of v'_{01} in \mathcal{Q}_n^{00} . Then u_{00} is neither deleted nor used even vertex in \mathcal{Q}_n^{00} (there are no deleted even vertices in \mathcal{Q}_n^{00}). There are $k - r - 2 + r + s - k + 1 = s - 1 \leq k - 3 \leq (n - 2) - 3$ used or deleted odd vertices in \mathcal{Q}_n^{00} . Therefore there exists an odd neighbor v_{00} of u_{00} in \mathcal{Q}_n^{00} which is neither deleted nor used vertex.

Let u_{10} be the even neighbor of v_{00} in \mathcal{Q}_n^{10} . Clearly u_{10} is neither used nor deleted vertex (there are no deleted even vertices in \mathcal{Q}_n^{10}). Let also v_{10} be an undeleted odd vertex in \mathcal{Q}_n^{10} which even neighbor u'_{00} in \mathcal{Q}_n^{00} is not an used vertex. There are $k - r - 2 + r + s - k + 1 = s - 1$ used even and $s - 1$ deleted odd vertices in \mathcal{Q}_n^{10} . Since $s - 1 \leq k - 3 \leq (n - 2) - 3$, it follows from (T) that there exists a Hamiltonian path γ_2 for \mathcal{Q}_n^{10} minus all deleted or used vertices that connects u_{10} to v_{10} .

Let u''_{00} be an unused even vertex in \mathcal{Q}_n^{00} different from u'_{00} , which odd neighbor v''_{01} in \mathcal{Q}_n^{01} is neither deleted nor used vertex. There are $k - r - 2 + r + s - k + 1 = s - 1$ used even and $k - r - 2 + r + s - k + 1 + 1 = s$ deleted or used odd vertices in \mathcal{Q}_n^{00} . Since $s - 1 \leq (k - 2) - 1 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{00} minus all deleted or used vertices that connects u'_{00} to u''_{00} .

There are $k - r + r + s - k = s$ deleted or used even and $k - r - 2 + r + s - k + 1 = s - 1$ used or deleted odd vertices in \mathcal{Q}_n^{01} (v_{01} is not counted). Since $s - 1 \leq (k - 2) - 1 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted or used vertices that connects v''_{01} to v_{01} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$\begin{aligned} v_{11} \xrightarrow{\gamma_1} u_{11} \rightarrow v'_{01} \rightarrow (u_{00}, v_{00}) \rightarrow u_{10} \xrightarrow{\gamma_2} v_{10} \rightarrow \\ u'_{00} \xrightarrow{\gamma_3} u''_{00} \rightarrow v''_{01} \xrightarrow{\gamma_4} v_{01}. \end{aligned}$$

(ii) $t \leq k - 1$ and therefore there is at least one deleted odd vertex in \mathcal{Q}_n^{01} , hence $p \leq 1$.

Take a model path μ in \mathcal{Q}_n^{01} that begins with an odd vertex $o_{01} = v_{01}$ and following μ make $r - t + s - 1$ short cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow o_{11} \rightarrow (e, o')_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Then begin with $o_{01} = a'_{01}$ and following μ make p short cycles of the type

$$o_{01} \rightarrow e_{00} \rightarrow (o, e)_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a''_{01} (if $p = 0$ we do not make such cycles).

Next, begin with $o_{01} = a''_{01}$ and following μ make $t - r - p - 1$ cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'''_{01} .

Finally, begin with $o_{01} = a'''_{01}$ and following μ , extend the resulting path with the following path with length four

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow o_{11}.$$

We denote the end vertex of the resulting path by v_{11} .

The total number of the constructed short cycles is $r - t + s - 1 + p + t - r - p - 1 + 1 = s - 1 \leq (k - 2) - 1 \leq k - 3$, hence the length of μ is enough for that construction.

Let v'_{11} be neither deleted nor used odd vertex in \mathcal{Q}_n^{11} different from v_{11} , which even neighbor u_{10} in \mathcal{Q}_n^{10} is not an used vertex. There are $p + t - r - p - 1 + r = t - 1$ used even and $r - t + s - 1 + p + t - r - p - 1 + t - s = t - 2$ used or deleted odd vertices in \mathcal{Q}_n^{11} . Since $t - 2 \leq (k - 1) - 1 = k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{11} minus all deleted or used vertices.

Let u'_{10} be an unused even vertex in \mathcal{Q}_n^{10} different from u_{10} , which odd neighbor v_{00} in \mathcal{Q}_n^{00} is neither deleted nor used vertex. There are $r - t + s - 1 + p + t - r - p - 1 + 1 = s - 1$ used even and $p + s - p = s$ used or deleted odd vertices in \mathcal{Q}_n^{10} . Since $s - 1 \leq k - 3 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_2 for \mathcal{Q}_n^{10} minus all deleted or used vertices that connects u_{10} to u'_{10} .

Let u_{00} be an unused even vertex in \mathcal{Q}_n^{00} which odd neighbor v'_{01} in \mathcal{Q}_n^{01} is neither deleted nor used vertex. There are $r - t + s - 1 + p + t - r - p - 1 + 1 = s - 1$ used even and $r - t + s - 1 + t - r - p - 1 + 1 + p = s - 1$ deleted or used odd vertices in \mathcal{Q}_n^{00} . Since $s - 1 \leq (k - 2) - 1 \leq (n - 2) - 3$, it follows from (T) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{00} minus all deleted or used vertices that connects v_{00} to u_{00} .

There are $r - t + s - 1 + k - r = k - t + s - 1$ deleted or used even and $r - t + s - 1 + p + t - r - p - 1 + k - t = k - t + s - 2$ used odd vertices in \mathcal{Q}_n^{01} (v_{01} is not counted). Since $t - s > 0$, we have $k - (t - s) - 2 \leq k - 3 \leq (n - 2) - 3$. Then it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted or used vertices that connects v'_{01} to v_{01} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} v'_{11} \rightarrow u_{10} \xrightarrow{\gamma_2} u'_{10} \rightarrow v_{00} \xrightarrow{\gamma_3} u_{00} \rightarrow v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

Case (G) $k \leq n - 3$ and there exists a column A in M_2 that separates the even vertices in the way $(r, k - r)$, where $2 \leq r \leq k - 2$, and the odd vertices in the way $(1, k - 1)$. Also, there exists a column B in M_o that separates the odd vertices in the way $(1, k - 1)$ but in a different way than A .

Since B separates the deleted odd vertices in the way $(1, k - 1)$ then, according Remark 3.1, B separates all deleted vertices in the way $(k + 1, k - 1)$.

We split \mathcal{Q}_n using A and B . Without loss of generality, we can assume that the deleted vertices are distributed as follows:

$$\{v_1\} \subset \mathcal{Q}_n^{00}, \{u_{r+1}, \dots, u_k\} \subset \mathcal{Q}_n^{01}, \{v_2, \dots, v_{k-1}\} \subset \mathcal{Q}_n^{10}, \text{ and} \\ \{u_1, \dots, u_r, v_k\} \subset \mathcal{Q}_n^{11}.$$

Take a model path μ in \mathcal{Q}_n^{01} that begins with an odd vertex $o_{01} = v_{01}$ and following μ make $k - r - 1$ short cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow (o, e)_{11} \rightarrow o'_{01}.$$

We denote the end vertex of the resulting path by a'_{01} .

Then begin with $o_{01} = a'_{01}$ and following μ make $r - 1$ short cycles of the type

$$o_{01} \rightarrow (e, o)_{00} \rightarrow e_{10} \rightarrow o_{11} \rightarrow (e, o')_{01}.$$

We denote the end vertex of the resulting path by a''_{01} .

Let the neighbor of a''_{01} in \mathcal{Q}_n^{00} be u_{00} . We extend the constructed path with the edge (a''_{01}, u_{00}) .

The total number of the constructed short cycles is $k - r - 1 + r - 1 = k - 2 \leq (n - 3) - 2$, hence the length of μ is enough for that construction.

There are $r + k - r - 1 = k - 1$ deleted or used even and $k - r - 1 + r - 1 + 1 = k - 1$ used or deleted odd vertices in \mathcal{Q}_n^{11} . Since

$k - 1 \leq (n - 3) - 1 = (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{11} minus all deleted or used vertices. Let u_{11} and v_{11} be two neighbors in γ' such that the odd neighbor v'_{01} of u_{11} in \mathcal{Q}_n^{01} is different from v_{01} and is neither deleted nor used vertex. Clearly the even neighbor u_{10} of v_{11} in \mathcal{Q}_n^{10} is also neither deleted nor used vertex. We denote by γ_3 the Hamiltonian path for \mathcal{Q}_n^{11} minus all deleted or used vertices that is defined by γ' and connects v_{11} to u_{11} .

Let u'_{00} be neither deleted nor used vertex in \mathcal{Q}_n^{00} different from u_{00} , which neighbor v_{10} in \mathcal{Q}_n^{10} is neither deleted nor used vertex. There are $k - r - 1 + r - 1 = k - 2$ used even and $k - r - 1 + r - 1 + 1 = k - 1$ used or deleted odd vertices in \mathcal{Q}_n^{00} . Since $k - 2 \leq (n - 3) - 2 = (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_1 for \mathcal{Q}_n^{00} minus all deleted or used vertices that connects u_{00} to u'_{00} .

There are $k - r - 1 + r - 1 = k - 2$ used even and $k - 2$ deleted odd vertices in \mathcal{Q}_n^{10} . Since $k - 2 \leq (n - 2) - 3$, it follows from (T) that there exists a Hamiltonian path γ_2 for \mathcal{Q}_n^{10} minus all deleted or used vertices that connects v_{10} to u_{10} .

There are $k - r + r - 1 = k - 1$ deleted or used even and $k - r - 1 + r - 1 = k - 2$ used odd vertices in \mathcal{Q}_n^{01} (v_{01} is not counted). Since $k - 2 \leq (n - 2) - 3$, it follows from (CG) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{01} minus all deleted or used vertices that connects v'_{01} to v_{01} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{00} \xrightarrow{\gamma_1} u'_{00} \rightarrow v_{10} \xrightarrow{\gamma_2} u_{10} \rightarrow v_{11} \xrightarrow{\gamma_3} u_{11} \rightarrow v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

Remark 3.3. *In the following cases we assume that every column in M_2 separates the even and the odd vertices in the way $(1, k - 1)$ and all deleted vertices in the way (k, k) .*

Case (H) There exists a column A in M_o that separates the deleted odd vertices in the way $(s, k - s)$, where $2 \leq s \leq k - 2$, and a column B in M_e that separates the deleted even vertices in the way $(r, k - r)$, where $2 \leq r \leq k - 2$.

We split \mathcal{Q}_n using A and B . Without loss of generality we can assume that $r \geq s$ and that the deleted vertices are distributed as follows:

$$\begin{aligned} \{v_1, v_2, \dots, v_s\} &\subset \mathcal{Q}_n^{00}, \{u_1, u_2, \dots, u_r\} \subset \mathcal{Q}_n^{11}, \text{ and} \\ \{v_{s+1}, \dots, v_k, u_{r+1}, \dots, u_k\} &\subset \mathcal{Q}_n^{01}. \end{aligned}$$

Take a model path μ in \mathcal{Q}_n^{10} that begins with an even vertex $e_{10} = u_{10}$ and following μ make $r - s$ short cycles of the type

$$e_{10} \rightarrow o_{11} \rightarrow e_{01} \rightarrow (o, e)_{00} \rightarrow (o, e')_{10}$$

(if $r - s = 0$ we do not make such cycles). We denote the end vertex of the resulting path by a'_{10} .

There are $k - s$ pairs of even and odd deleted or used vertices in \mathcal{Q}_n^{01} . Since $2 \leq s \leq k - 2$, we have $2 \leq k - s \leq k - 2 \leq (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle μ' for \mathcal{Q}_n^{01} minus all deleted or used vertices. Let μ'' be the projection of μ' on \mathcal{Q}_n^{10} . Clearly, a'_{10} belongs to μ'' .

Now begin with $e_{10} = a'_{10}$ and following μ'' (and μ'), continue with $s - 1$ cycles of the type

$$e_{10} \rightarrow o_{11} \rightarrow (e, o)_{01} \rightarrow e_{00} \rightarrow (o, e')_{10}.$$

We denote the end vertex of the resulting path by a''_{10} .

Let the neighbor of a''_{10} in \mathcal{Q}_n^{11} be v_{11} . We extend the constructed path with the edge (a''_{10}, v_{11}) .

Since we have been following μ' , $2(s - 1)$ consecutive vertices from μ' have been used in these short cycles for all edges of the type $(e, o)_{01}$. The length of μ' is $2^{n-2} - 2(k - s)$ and since $2^k > 4k$ for $k \geq 5$ we have

$$2^{n-2} - 2(k - s) > 2^k - 2k > 2k > 2(s - 1).$$

Therefore what remains unused from γ_1 forms a path γ_2 . Notice that the end vertices of γ_2 have different parity. We denote these end vertices by u_{01} and v_{01} .

Denote the neighbor of u_{01} in \mathcal{Q}_n^{11} by v'_{11} . Clearly v'_{11} has not been used so far and is not a deleted vertex. There are r deleted even vertices and $r - s + s - 1 = r - 1$ used odd vertices in \mathcal{Q}_n^{11} . Since $r - 1 \leq k - 3$ we can use (CG) to find a Hamiltonian path γ_1 for \mathcal{Q}_n^{11} minus all deleted or used vertices that connects v_{11} to v'_{11} .

Denote the neighbor of v_{01} in \mathcal{Q}_n^{00} by u_{00} and let u'_{00} be any other undeleted and unused even vertex in \mathcal{Q}_n^{00} . Then its neighbor v_{10} in \mathcal{Q}_n^{10} has not been used so far. There are $r - s + s = r$ deleted or used odd vertices and $r - s + s - 1 = r - 1$ used even vertices in \mathcal{Q}_n^{00} . Since $r - 1 \leq k - 3$ we can use (CG) to find a Hamiltonian path γ_3 for \mathcal{Q}_n^{00} minus all deleted and used vertices that connects u_{00} to u'_{00} .

There are $r - s + s - 1 = r - 1$ pairs of used even and odd vertices in \mathcal{Q}_n^{10} . Since $r - 1 \leq k - 3$ we can use (T) to find a Hamiltonian path γ_4 for \mathcal{Q}_n^{10} minus all used vertices that connects v_{10} to u_{10} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} v'_{11} \rightarrow u_{01} \xrightarrow{\gamma_2} v_{01} \rightarrow u_{00} \xrightarrow{\gamma_3} u'_{00} \rightarrow v_{10} \xrightarrow{\gamma_4} u_{10}.$$

Remark 3.4. *In all cases below we assume that $k \leq n - 3$ and that every coordinate that separates only the deleted odd vertices separates them in the way $(1, k - 1)$, and therefore it separates all vertices in the way $(k + 1, k - 1)$.*

(I) $k \leq n - 3$, there is a column A in M_o and there is a column B in M_e .

We split \mathcal{Q}_n using A and B . Without loss of generality, we can assume that the deleted vertices are distributed as follows:

$$\begin{aligned} \{v_1, v_2, \dots, v_s\} &\subset \mathcal{Q}_n^{00}, \{u_1, u_2, \dots, u_{k-1}\} \subset \mathcal{Q}_n^{11}, \text{ and} \\ \{v_{s+1}, \dots, v_k, u_k\} &\subset \mathcal{Q}_n^{01}. \end{aligned}$$

Take a model path μ in \mathcal{Q}_n^{10} that begins with an even vertex $e_{10} = u_{10}$ and following μ make $k - 1 - s$ short cycles of the type

$$e_{10} \rightarrow o_{11} \rightarrow e_{01} \rightarrow (o, e)_{00} \rightarrow (o, e')_{10}$$

(if $k - 1 - s = 0$ we do not make such cycles). We denote the end vertex of the resulting path by a'_{10} .

Then begin with $e_{10} = a'_{10}$ and following μ make $s - 1$ cycles of the type

$$e_{10} \rightarrow o_{11} \rightarrow (e, o)_{01} \rightarrow e_{00} \rightarrow (o, e')_{10}.$$

We denote the end vertex of the resulting path by a''_{10} .

Let the neighbor of a''_{10} in \mathcal{Q}_n^{11} be v_{11} . We extend the constructed path with the edge (a''_{10}, v_{11}) .

The total number of the constructed short cycles is $k - 1 - s + s - 1 = k - 2 \leq (n - 3) - 2$, hence the length of μ is enough for that construction.

There are $k - s - 1 + s - 1 + 1 = k - 1$ deleted or used even and $k - s + s - 1 = k - 1$ used or deleted odd vertices in \mathcal{Q}_n^{01} . Since $k - 1 \leq (n - 3) - 1 = (n - 2) - 2$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{01} minus all deleted or used vertices. Let u_{01} and v_{01} be two neighbors in γ' such that the odd neighbor v'_{11} of u_{01} in \mathcal{Q}_n^{11} is different from v_{11} and is neither deleted nor used vertex and the

even neighbor u_{00} of v_{01} in \mathcal{Q}_n^{00} is also neither deleted nor used vertex. We denote by γ_2 the Hamiltonian path for \mathcal{Q}_n^{01} minus all deleted or used vertices that is defined by γ' and connects u_{01} to v_{01} .

There are $k - 1$ deleted even vertices and $k - s - 1 + s - 1 = k - 2$ used odd vertices in \mathcal{Q}_n^{11} . Since $k - 2 \leq (n - 3) - 2 = (n - 2) - 3$ we can use (CG) to find a Hamiltonian path γ_1 for \mathcal{Q}_n^{11} minus all deleted and used vertices that connects v_{11} to v'_{11} .

Let u'_{00} be any undeleted and unused odd vertex in \mathcal{Q}_n^{00} different from u_{00} . Then its neighbor v_{10} in \mathcal{Q}_n^{10} has not been used so far. There are $k - s - 1 + s - 1 = k - 2$ used even vertices and $k - s - 1 + s = k - 1$ deleted or used odd vertices in \mathcal{Q}_n^{00} . Since $k - 2 \leq (n - 3) - 2 = (n - 2) - 3$ we can use (CG) to find a Hamiltonian path γ_3 for \mathcal{Q}_n^{00} minus all deleted and used vertices that connects u_{00} to u'_{00} .

There are $k - s - 1 + s - 1 = k - 2$ pairs of used even and odd vertices in \mathcal{Q}_n^{10} . Since $k - 2 \leq (n - 2) - 3$ we can use (T) to find a Hamiltonian path γ_4 for \mathcal{Q}_n^{10} minus all used vertices that connects v_{10} to u_{10} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} v'_{11} \rightarrow u_{01} \xrightarrow{\gamma_2} v_{01} \rightarrow u_{00} \xrightarrow{\gamma_3} u'_{00} \rightarrow v_{10} \xrightarrow{\gamma_4} u_{10}.$$

(J) $k \leq n - 3$, there is a column A in M_o and there is a column B in M_2 that separates the odd vertices in different way than A .

It follows from our hypotheses that A separates the deleted odd vertices in the way $(1, k - 1)$ and all vertices in the way $(k + 1, k - 1)$. Also, B separates the odd and the even vertices in the way $(1, k - 1)$ and all vertices in the way (k, k) . Therefore, without loss of generality, we can assume that the deleted vertices are distributed as follows:

$$\{v_1\} \subset \mathcal{Q}_n^{00}, \{v_3, \dots, v_k\} \subset \mathcal{Q}_n^{10}, \{u_1, v_2\} \subset \mathcal{Q}_n^{11}, \text{ and} \\ \{u_2, \dots, u_k\} \subset \mathcal{Q}_n^{01}.$$

Take a model path μ in \mathcal{Q}_n^{10} that begins with an even vertex $e_{10} = u_{10}$ and following μ make $k - 2$ short cycles of the type

$$e_{10} \rightarrow (o, e)_{11} \rightarrow o_{01} \rightarrow (e, o)_{00} \rightarrow e'_{10}.$$

We denote the end vertex of the resulting path by u'_{10} . Let the neighbor of u'_{10} in \mathcal{Q}_n^{11} be v_{11} . We extend the constructed path with the edge (u'_{10}, v_{11}) .

The total number of the constructed short cycles is $k - 2 \leq (n - 3) - 2$, hence the length of μ is enough for that construction.

There are $k - s + s - 1 = k - 1$ pairs of even and odd deleted or used vertices in \mathcal{Q}_n^{01} . Since $k - 1 \leq (n - 3) - 1$, it follows from (L) that there exists a Hamiltonian cycle γ' for \mathcal{Q}_n^{01} minus all deleted or used vertices.

There are $k - 2 + 1 = k - 1 \leq (n - 2) - 2$ pairs of deleted or used even and odd vertices in \mathcal{Q}_n^{11} . Therefore, according to (L), there exists a Hamiltonian cycle for \mathcal{Q}_n^{11} minus all deleted or used vertices. This cycle contains v_{11} . Let u_{11} be a neighbor of v_{11} in that cycle and let γ_1 be the Hamiltonian path for \mathcal{Q}_n^{11} minus all deleted or used vertices determined by this cycle that connects v_{11} to u_{11} .

The neighbor v_{01} of u_{11} is clearly neither deleted nor used odd vertex in \mathcal{Q}_n^{01} . Let $v'_{01} \neq v_{01}$ be any unused odd vertex in \mathcal{Q}_n^{01} . There are $k - 1$ deleted even vertices and $k - 2 \leq (n - 2) - 3$ used odd vertices in \mathcal{Q}_n^{01} . Then it follows from (CG) that there exists a Hamiltonian path γ_2 for \mathcal{Q}_n^{01} minus all deleted or used vertices that connects v_{01} to v'_{01} .

The neighbor u_{00} of v_{01} is clearly neither deleted nor used even vertex in \mathcal{Q}_n^{00} . Let $u'_{00} \neq u_{00}$ be any unused even vertex in \mathcal{Q}_n^{00} which neighbor v_{10} is not a deleted even vertex in \mathcal{Q}_n^{10} . There are $k - 2 \leq (n - 2) - 3$ used even vertices and $k - 2 + 1 = k - 1$ deleted or used even vertices in \mathcal{Q}_n^{00} . Then it follows from (CG) that there exists a Hamiltonian path γ_3 for \mathcal{Q}_n^{00} minus all deleted or used vertices that connects u_{00} to u'_{00} .

There are $k - 2$ deleted odd and $k - 2 \leq (n - 2) - 3$ used even vertices in \mathcal{Q}_n^{10} . Then it follows from (T) that there exists a Hamiltonian path γ_4 for \mathcal{Q}_n^{10} minus all deleted or used vertices that connects v_{10} to u_{10} .

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} u_{11} \rightarrow v_{01} \xrightarrow{\gamma_2} v'_{01} \rightarrow u_{00} \xrightarrow{\gamma_3} u'_{00} \rightarrow v_{10} \xrightarrow{\gamma_4} u_{10}.$$

The proof is completed.

REFERENCES

- [CG1] N. Castañeda and I. S. Gotchev, *Path coverings with prescribed ends in faulty hypercubes*, submitted for publication (2007).
- [CG2] N. Castañeda and I. S. Gotchev, *Embedded cycles in faulty hypercubes*, Journal of Combinatorial Optimization (2009), in press; DOI: 10.1007/s10878-008-9205-6.
- [CGGL1] N. Castañeda, V. Gochev, I. Gotchev and F. Latour, *Hamiltonian laceability of hypercubes with faults of charge one*, to appear in Congressus Numerantium (2009).
- [CGGL2] N. Castañeda, V. Gochev, I. Gotchev and F. Latour, *Proof of Locke's conjecture, II*, in preparation.

- [L] S. C. Locke, *Problem 10892*, The American Mathematical Monthly, **108** (2001), 668.
- [LS] S. C. Locke and R. Stong, *Spanning cycles in hypercubes*, The American Mathematical Monthly, **110**, (2003), 440–441.
- [LW] M. Lewinter and W. Widulski, *Hyper-Hamilton laceable and caterpillar-spannable product graphs*, Comput. Math. Appl. **34** (1997), 99 – 104.

DEPARTMENT OF MATHEMATICAL SCIENCES, CENTRAL CONNECTICUT STATE UNIVERSITY, 1615 STANLEY STREET, NEW BRITAIN, CT 06050

E-mail address: `castanedan@ccsu.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, CENTRAL CONNECTICUT STATE UNIVERSITY, 1615 STANLEY STREET, NEW BRITAIN, CT 06050

E-mail address: `gotchevi@ccsu.edu`