Sequentially Closed and Absolutely Closed Spaces.

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Sunto. - *Si introduce il concetto di spazio sequenzialmente $\mathcal{T}$-chiuso. Si dimostra che per alcune categorie $\mathcal{T}$ definite tramite successioni, questi spazi sono strettamente legati sia agli spazi numerabilmente compatti e spazi sequenzialmente compatti, sia agli spazi assolutamente $\mathcal{T}$-chiusi nel senso di* [5] *e* [7].

0. - Introduction.

Let $\mathcal{T}$ be a category of topological spaces, a $\mathcal{T}$ space X is said to be *sequentially* (*absolutely*) *$\mathcal{T}$-closed* if it is sequentially closed (equalizer of two continuous maps into a $\mathcal{T}$ space) in every $\mathcal{T}$-space Y in which it is embedded. We remind here that the equalizer of two maps $f, g\colon Y \to Z$ is the subspace of Y with the subset $\{y \in Y\colon f(y) = g(y)\}$ as underying set. It is clear that if $\mathcal{T}$ is contained in the category Haus of Hausdorff spaces, then every absolutely $\mathcal{T}$-closed space is $\mathcal{T}$-closed since every equalizer in $\mathcal{T}$ is a closed subspace.

The concept of an absolutely $\mathcal{T}$-closed space was introduced by Giuli and the first named author in [7] and developed later in [5] and [8]. While for $\mathcal{T} \subset$ Haus the $\mathcal{T}$-closed spaces abound and are well studied (see [2]), for $\mathcal{T} \not\subset$ Haus the $\mathcal{T}$-closed spaces are quite rare. For example no T_0-closed spaces exist and every T_1-closed space is finite ([1]), other examples are given in § 2 of the present paper. This is why the absolutely $\mathcal{T}$-closed spaces can be considered as a substitute of the $\mathcal{T}$-closed spaces in this case. In fact, for all categories $\mathcal{T}$ of weakly Hausdorff spaces considered in [4], [9], [11] and [12] the absolute $\mathcal{T}$-closedness implies the $\mathcal{T}$-closedness since for these categories $\mathcal{T}$ every closed subspace is an equalizer in $\mathcal{T}$. On the other hand the definition of an absolutely $\mathcal{T}$-closed space is purely categorical and this permits to define analogously absolutely $\mathcal{T}$-closed objects in various (also abstract) categories $\mathcal{T}$. For example absolutely closed semigroups and algebras were introduced by Isbell in 1965, more about absolutely closed objects can be found in [8]. Absolutely $\mathcal{F}$-closed spaces for some categories $\mathcal{F} \subset$ Haus were studied in [5].

In this paper we introduce the concept of a sequentially closed space and we show that these spaces are also a good substitute of the $\mathcal{F}$-closed spaces to certain extent. Let US (SUS) denote the category of all topological spaces in which every convergent sequence has a unique limit (cluster) point. It is easy to see that if $\mathcal{F} \subset$ US ($\mathcal{F} \subset$ SUS) then every sequentially compact (countably compact) $\mathcal{F}$-space is sequentially $\mathcal{F}$-closed. In § 1 we show that the converse is also true, i.e. sequential US-closedness implies sequential compactness and sequential SUS-closedness implies countable compactness. On the other hand in § 2 we show that every US-closed space is finite and every SUS-closed space is a finite union of converging sequences. Therefore for $\mathcal{F} =$ US and SUS the sequential $\mathcal{F}$-closedness seems to be a really good substitute of the $\mathcal{F}$-closedness.

Tozzi [12] proved that the equalizers in SUS are precisely the sequentially closed subspaces, this is why the sequentially SUS-closed spaces coincide also with the absolutely SUS-closed spaces. On the other hand every absolutely US-closed spaces is sequentially compact, while the converse is not true in general (see § 2). We show that every countable sequentially compact US-space X is absolutely US-closed and conjecture that this remains true whenever $|X| < 2^{\aleph_0}$.

In a subsequent paper [10] the second named author continues the study of sequentially $\mathcal{F}$-closed spaces for $\mathcal{F} \subset$ Haus.

It is a pleasure to thank E. Giuli for many stimulating discussions on the absolutely $\mathcal{F}$-closed spaces during his stay in Sofia in 1985.

1. – Sequentially $\mathcal{F}$-closed spaces.

We denote by N the (discrete) set of naturals. A sequence in a topological space X will be considered as a map $S: \mathrm{N} \to X$. Let MON denote the set of all monotone one-to-one maps $\mathrm{N} \to \mathrm{N}$, then a subsequence of S is a composition $S \circ S'$ for some $S' \in$ MON. For a sequence S in X we denote by $\lim S$ the set of limit points of S. Clearly $X \in$ US if $|\lim S| \leq 1$ for every sequence S in X, where $|M|$ denotes the cardinality of a set M. For a subset M of a topological space X we denote by $\bar{M}$ the closure of M and by $[M]_S$ the sequential closure of M; i.e. $[M]_S = \{x \in X : x \in \lim S$ for some sequence S in $M\}$. For a sequence S we denote by $S(\mathrm{N})$ the range of S.

Let A be a countably infinite subset of a topological space X. The open filter on X having as a base all open sets in X containing all but a finite number of the elements of A will be called *elementary open filter generated by* A. It will be shown below that if $X \in \mathrm{US}$ then two countable subsets A' and A'' of X generate the same open elementary filter iff the symmetric difference

$$(A' \setminus A'') \cup (A'' \setminus A')$$

is finite (the sufficiency is obvious). Sometimes we shall avoid mentioning explicitely the countable set generating such a filter $\mathcal{F}$, in such a case we refer to $\mathcal{F}$ as an open elementary filter. These filters are introduced by the second named author in [10] following the definition of elementary filter in Bourbaki [3], chap. 1, § 6, 8°.

For an open filter $\mathcal{F}$ on a topological space X we denote by $X_{\mathcal{F}}$ the standard one-point extension of X by means of $\mathcal{F}$. This is the set $X \cup \{\mathcal{F}\}$ provided with a topology which makes X open in $X_{\mathcal{F}}$, the relative topology of X coincides with the original topology of X and a *nbd* of the point $\{\mathcal{F}\}$ has the form $\{\mathcal{F}\} \cup F_\alpha$, where $F_\alpha \subset X$ and $F_\alpha \in \mathcal{F}$. If X is T_1, then $X_{\mathcal{F}}$ is T_1 iff

$$\cap \{U\colon U \in \mathcal{F}\} = \emptyset\,.$$

On the other hand $[X]_S = X_{\mathcal{F}}$ iff $\mathcal{F}$ is contained in some elementary open filter on X. If $X \in \mathrm{US}$ and $\mathcal{F}$ is an elementary open filter on X, then every sequence in X converging to $\{\mathcal{F}\}$ in X is eventually in the countable subset of X generating $\mathcal{F}$ (see lemma 1.2 below).

LEMMA 1.1. – *Let* X *be a* T_1 *space and let* A *be a countable subset of* X. *Then for every non-negative integer* m *and for every sequence* S *in* X *satisfying* $|[S(\mathrm{N})]_s \cap A| \leq m$ *there exists a subsequence* S' *of* S *such that* $|\overline{S'(\mathrm{N})} \cap A| \leq m$.

PROOF. – Let S and m satisfy the hypothesis, then there exists a subset B of A such that $|A \setminus B| \leq m$ and

$$[S(\mathrm{N})]_s \cap B = \emptyset\,. \tag{1}$$

If A is finite there is nothing to prove, so assume that A is infinite, then B is also infinite. Let $x_1, x_2, \ldots, x_n, \ldots$ be an ordering of B. Since $x_1 \notin \lim S$ by (1) there exist an open *nbd* U_1 of x_1 and a subsequence $S \circ S_1 (S_1 \in \mathrm{MON})$ such that $U_1 \cap (S \circ S_1(\mathrm{N})) = \emptyset$. Since $x_2 \notin \lim S \circ S_1$ by (1) there exist an open *nbd* U_2 of x_2 and a subsequence of $S \circ S_1$ defined by $S_2 \in \mathrm{MON}$ such that $U_2 \cap (S \circ S_1 \circ \circ S_2(\mathrm{N})) = \emptyset$. In this way we find inductively a sequence of open sets $U_1, U_2, \ldots, U_n, \ldots$ and $S_n \in \mathbf{MON}$, such that

$$(2) \qquad x_n \in U_n \text{ and } U_n \cap \big(S \circ S_1 \circ S_2 \circ \ldots \circ S_n(\mathrm{N})\big) = \emptyset \text{ for } n = 1, 2, \ldots .$$

Set

$$S'(n) = S \circ S_1 \circ \ldots \circ S_n(n)$$

for $n = 1, 2, \ldots$, then S' is a subsequence of S with the desired properties. In fact for every $x_n \in B$, $U_n \cap \big(S'(\mathrm{N})\big) \subset U_n \cap \{S'(i): i = 1, 2, \ldots, n-1\}$ because of (2). Since X is T_1 and the latter set is finite there exists a *nbd* U'_n of x_n which avoids the sequence S'. Thus $S'(\mathrm{N}) \cup B = \emptyset$, so $|\overline{S'(\mathrm{N})} \cap A| \leq m$. Q.E.D.

LEMMA 1.2. – *Let $X \in \mathrm{US}$, A be a countably infinite subset of X and $\mathcal{F}$ be the open elementary filter generated by A. Then for every sequence S in X $\{\mathcal{F}\} \in \lim S$ in $X_{\mathcal{F}}$ iff S is eventually in A, i.e. $S(\mathrm{N}) \setminus A$ is finite.*

PROOF. – The sufficiency is obvious since for every ordering $a_1, a_2, \ldots, a_n, \ldots$ of A the sequence $\{a_n\}$ converges to $\{\mathcal{F}\}$ in $X_{\mathcal{F}}$.

Now assume that S is a sequence in X with $\{\mathcal{F}\} \in \lim S$ and $S(\mathrm{N}) \setminus A$ is infinite. We can assume wlog that $S(\mathrm{N}) \cap A = \emptyset$. Let $S' \in \mathbf{MON}$ be such that $S \circ S'$ is convergent in X, if such S' exists, otherwise $S' = id_{\mathrm{N}}$. Now by $X \in \mathrm{US}$ $|[S \circ S'(\mathrm{N})]_s \cap A| \leq 1$, so by the previous lemma there exists $S'' \in \mathbf{MON}$ such that $|\overline{S \circ S' \circ S''(\mathrm{N})} \cap A| \leq 1$. Then $\{\mathcal{F}\} \cup \big(X \setminus \overline{S \circ S' \circ S''(\mathrm{N})}\big)$ is an open *nbd* of $\{\mathcal{F}\}$ in $X_{\mathcal{F}}$ which avoids the subsequence $S \circ S' \circ S''$ of S—a contradiction. Q.E.D.

The last lemma shows that in a US-space every open elementary filter determines its generating countable set up to a finite subset.

THEOREM 1.3. – *An* US*-space is sequentially* US*-closed iff it is sequentially compact.*

PROOF. – The sufficiency is obvious, so we prove the necessity. Assume that X is a US-space which is not sequentially compact. Then there exists a sequence S in X with no convergent subsequences. Denote by $\mathcal{F}$ the open elementary filter generated by $A = S(\mathrm{N})$ in X. Then X is not sequentially closed in its extension $X_{\mathcal{F}}$, so it suffices to show that $X_{\mathcal{F}} \in \mathrm{US}$ this will prove that X is not sequentially US-closed.

Let S' be a convergent sequence in $X_{\mathcal{F}}$. Consider first the case $\{\mathcal{F}\} \in \lim S'$. If S' coincides definitely with the constant sequence $\{\mathcal{F}\}$ then clearly S' has no limit points in X since $X_{\mathcal{F}}$ is T_1. Otherwise $S'(\mathrm{N}) \cap X$ gives a subsequence of S which by lemma 1.2 is eventually in A, so it has no convergent subsequences in X. Therefore S' has no other limit point in $X_{\mathcal{F}}$.

If $\{\mathcal{F}\} \notin \lim S'$, then S' is eventually in X since for every $x \in \lim S'$ X is an open *nbd.* Now $X \in \mathrm{US}$ implies that S' has a unique limit point. This proves that $X_{\mathcal{F}} \in \mathrm{US}$. Q.E.D.

Here follows the counterpart for SUS.

THEOREM 1.4. – *Let X be an* SUS-*space. Then X is sequentially* SUS-*closed iff X is countably compact.*

PROOF. – The sufficiency is obvious, so we prove the necessity.

It is enough to show that if X is not countably compact then X is not sequentially SUS-closed. Let A be a countably infinite subset of X with no cluster points in X. Denote by $\mathcal{F}$ the elementary open filter generated by A in X. As in the above proof it suffices to show that $X_{\mathcal{F}} \in \mathrm{SUS}$.

Let S be a convergent sequence in $X_{\mathcal{F}}$. If $\{\mathcal{F}\} \notin \lim S$ then S is definitely in X, so it has a unique cluster point in X since $X \in \mathrm{SUS}$. Let us see that $\{\mathcal{F}\}$ is not a cluster point of S in $X_{\mathcal{F}}$. The argument applied in the above proof gives $X_{\mathcal{F}} \in \mathrm{US}$, thus no subsequence of S converges to $\{\mathcal{F}\}$. In particular $S(\mathbb{N}) \cap A$ is finite. Since $X \in \mathrm{SUS}$ and S is convergent in X, it follows that $\overline{S(\mathbb{N})} \cap A$ is finite too. Now $U = X \setminus \overline{S(\mathbb{N})}$ is open and $A \setminus U$ is finite, so $U \in \mathcal{F}$. Therefore $\{\mathcal{F}\}$ is not a cluster point of S, so S has a unique cluster point in $X_{\mathcal{F}}$.

Assume that $\{\mathcal{F}\} \in \lim S$, then by lemma 1.2 it follows that the sequence S is eventually in the subset A of X. Since A has no cluster points in X the proof is finished. Q.E.D.

Now we give four more examples.

EXAMPLE 1.5. – (a) The sequentially T_1-closed spaces are finite. In fact for every infinite T_1 space X the filter $\mathcal{F}$ of cofinite subset of X gives a T_1 extension of $X_{\mathcal{F}}$ with $[X]_S = X_{\mathcal{F}}$.

(b) Let $\mathfrak{m} \geq \aleph_0$ be a cardinal number and let $X_{\mathfrak{m}}$ be the space provided with the cofinite topology and having cardinality $\mathfrak{m}$. Denote by $\mathfrak{C}_{\mathfrak{m}}$ the category of topological spaces X such that every continuous map $f\colon X_{\mathfrak{m}} \to X$ is constant ([11]). In fact $\mathfrak{C}_{\mathfrak{m}}$ coincides with those T_1 spaces which do not contain copies of $X_{\mathfrak{m}}$ (Th. 2.7, [11]). The same argument as above shows that for each cardinal $\mathfrak{m}$ the sequentially $\mathfrak{C}_{\mathfrak{m}}$-closed spaces are precisely the finite T_1 spaces.

(c) Let $\mathfrak{m}$ be as in (b) and $\mathfrak{T}_{\mathfrak{m}}$ denote the category of topological spaces in which every subspace of cardinality at most $\mathfrak{m}$ is

Hausdorff. It is easy to see that $X \in \mathfrak{T}_\mathfrak{m}$ is $\mathfrak{T}_\mathfrak{m}$-closed iff for every open filter $\mathcal{F}$ on X there exists a subset M of X and $|M| \leq \mathfrak{m}$, such that the filter $\mathcal{F}|_M$ generated by the intersections $M \cap F$, where $F \in \mathcal{F}$, has an adherent point in X. Now it is clear that X is sequentially $\mathfrak{T}_\mathfrak{m}$-closed iff for every open filter $\mathcal{F}$ on X which is contained in some elementary open filter on X there exists a subset M of X with $|M| \leq \mathfrak{m}$ such that $\mathcal{F}|_M$ has adherent points in X.

(*d*) Let $\mathfrak{m}$ be as in (*b*), intersections of families of cardinality at most $\mathfrak{m}$ of open subsets of a topological space will be called $G_\mathfrak{m}$ sets. In particular the $G_{\aleph_0}$ sets are the well known G_δ sets. Denote by $\mathfrak{D}_\mathfrak{m}$ the category of topological spaces in which distinct points can be separated by disjoint $G_\mathfrak{m}$ sets (see [6]). Obviously $X \in \mathfrak{D}_\mathfrak{m}$ is $\mathfrak{D}_\mathfrak{m}$-closed (sequentially $\mathfrak{D}_\mathfrak{m}$-closed) iff for every open filter $\mathcal{F}$ (contained in some elementary open filter on X) there exists a point x in X such that for every family $\{F_\alpha\}_{\alpha \in A}$ of members of $\mathcal{F}$ with $|A| \leq \mathfrak{m}$ and for every $G_\mathfrak{m}$ set U in X containing x $U \cap \bigcap \{F_\alpha : \hbar \in A\} \neq \emptyset$.

2. – Absolutely $\mathfrak{T}$-closed spaces for $\mathfrak{T} =$ US and SUS.

The absolutely $\mathfrak{T}$-closed spaces in various categories $\mathfrak{T}$ are studied in [8], here we stress on the case $\mathfrak{T} =$ US and SUS because of its relation to sequentially $\mathfrak{T}$-closed spaces.

The next theorem follows directly from theorem 1.4 and the definition of absolutely SUS-closed spaces (see the introduction).

THEOREM 2.1. – *For a space* $X \in$ SUS *the following three conditions are equivalent*:

a) X *is absolutely* SUS-*closed*;

b) X *is sequentially* SUS-*closed*;

c) X *is countably compact.*

For a subset M of a topological space X denote by $\boldsymbol{M}$ the set of points $x \in X$ such that there exists a sequence S in X with $x \in \lim S$ such that for every $S' \in$ MON $x \in \overline{S \circ S'(\mathrm{N}) \cap M}$. It was proved by Tozzi [12] (and in a more general situation in [4]), that if $X \in$ US and $M \subset X$, then M is an equalizer of two continuous maps into a US-space iff $M = \boldsymbol{M}$.

LEMMA 2.2. – *Every absolutely* US-*closed space is sequentially* US-*closed, so sequentially compact.*

PROOF. – It is enough to remark that always $[M]_S \subset \mathbf{M}$ holds. Q.E.D.

In the opposite direction the following can be proved.

PROPOSITION 2.3. – *Every countable sequentially compact* US-*space is absolutely* US-*closed.*

PROOF. – Let $X = \{x_1, x_2, \dots, x_n, \dots\}$ be a sequentially compact US-space. Assume that X is not absolutely US-closed. Then there exists a US-space Y containing X as a subspace such that $\mathbf{X} \neq X$. Let y be a point in $\mathbf{X} \setminus X$, then there exists a sequence S in Y converging to y such that $y \in \overline{S \circ S'(\mathbb{N}) \cap X}$ for every $S' \in \mathrm{MON}$. By $Y \in \mathrm{US}$ $x_1 \notin \lim S$, so there exist a *nbd* U_1 of x_1 and $S_1 \in \mathrm{MON}$ such that $S \circ S_1(\mathbb{N}) \cap U_1 = \emptyset$. Again by $Y \in \mathrm{US}$ $x_2 \notin \lim S \circ S_1$, so there exist a *nbd* U_2 of x_2 and $S_2 \in \mathrm{MON}$ such that $S \circ S_1 \circ S_2(\mathbb{N}) \cap \cap U_2 = \emptyset$. We find inductively open sets $U_1, U_2, \dots, U_n, \dots$ and $S_n \in \mathrm{MON}$ such that $x_n \in U_n$ and $S \circ S_1 \circ S_2 \circ \dots \circ S_n(\mathbb{N}) \cup U_n = \emptyset$ for every $n = 1, 2, \dots$ Define S' by $S'(n) = S \circ S_1 \circ S_2 \circ \dots \circ S_n(n)$, then for every $n = 1, 2, \dots, S'(\mathbb{N}) \cap U_n$ is finite. Since Y is T_1 this proves that $\overline{S'(\mathbb{N})} \cap X = S'(\mathbb{N}) \cap X$. Assume that this intersection is infinite, then by the sequential compactness of X S' has a convergent subsequence in X which contradicts $Y \in \mathrm{US}$.

The above argument supplied a subsequence S' of S such that $\overline{S'(\mathbb{N})} \cup X$ is finite, this contradicts the initial property of S. Therefore X is absolutely US-closed. Q.E.D.

The next proposition shows that the converse in lemma 2.2 is not true in general.

PROPOSITION 2.4. – *A space* $X \in \mathrm{SUS}$ *is not absolutely* US-*closed whenever the following two conditions are satisfied*:

(i) $|X| \geqq 2^{\aleph_0}$;

ii) *every point of* X *has a nbd* U *with* $|U| < 2^{\aleph_0}$.

PROOF. – By $d(Y)$ we denote the density of a topological space Y. It is well known that $d(\beta\mathbb{N} \setminus \mathbb{N}) = 2^{\aleph_0}$, this is why there exists a (discontinuous) map $f\colon X \to \beta\mathbb{N} \setminus \mathbb{N}$, such that $f(X)$ is dense in $\beta\mathbb{N} \setminus \mathbb{N}$. Now define a space $Y = \{y\} \cup \mathbb{N} \cup X$ with the following topology. The *nbds* of y are of the form $\{y\} \cup D \cup U$, where D is a cofinite subset of $\mathbb{N}$ and U is a subset of X whose complement is contained in a finite union of ranges of convergent sequences in X; the points of $\mathbb{N}$ are isolated and the basic *nbds* of the points x of X are of the form $U \cup V$, where U is an open subset of X containing x and V is a subset of $\mathbb{N}$ such that $V\in f(y)$ for every $y\in U$ (the points of $\beta\mathbb{N}\setminus\mathbb{N}$ will be identified with ultrafilters in $\mathbb{N}$). Clearly for every $D\subset\mathbb{N}$ and $x\in X$, $x\in\overline{D}$ in Y iff $f(x)\in\overline{D}^{\beta\mathbb{N}}$.

We are going to show that $Y\in\mathrm{US}$ and $y\in\mathbf{X}$ in Y which will prove that X is not absolutely US-closed.

Denote by S_0 the sequence in X defined by the identity of $\mathbb{N}$. To establish $Y\in\mathrm{US}$ it is enough to show that no subsequence S of S_0 converges to points x of X. In fact by ii) there exists an open *nbd* U of x in X with $|U|<2^{\aleph_0}$. Let $Z=\overline{S(\mathbb{N})}^{\beta\mathbb{N}}\setminus\mathbb{N}$, since $d(Z)=2^{\aleph_0}$, it follows that $f(U)\cap Z$ is not dense in Z. Hence there exists a non-empty open set W in Z with $W\cap f(U)=\emptyset$. We can assume wlog that $W=\overline{S'(\mathbb{N})}^{\beta\mathbb{N}}\setminus\mathbb{N}$, for some subsequence S' of S. For every $y\in U$ there exists a subset V_y of $\mathbb{N}$ such that $V_y\in f(y)$ and $V_y\cap S'(\mathbb{N})=\emptyset$. Then $V=\{V_y\colon y\in U\}$ is a subset of $\mathbb{N}$ such that $U\cup V$ is a *nbd* of x in Y which avoids the subsequence S' of S. Therefore S does not converge to x.

Let us prove finally that $y\in\mathbf{X}$ in Y. Obviously $y=\lim S_0$. Let S be a subsequence of S_0, then $Z=\overline{S(\mathbb{N})}^{\beta\mathbb{N}}\setminus\mathbb{N}$ is homeomorphic to $\beta\mathbb{N}\setminus\mathbb{N}$, so $d(Z)=2^{\aleph_0}$. On the other hand Z is open in $\beta\mathbb{N}\setminus\mathbb{N}$, so $Z\cap f(X)$ is dense in Z, hence $|Z\cap f(X)|\geq 2^{\aleph_0}$. As we remarked above $f(x)\in Z$ implies $x\in\overline{S(\mathbb{N})}$, so $|\overline{S(\mathbb{N})}\cap X|\geq 2^{\aleph_0}$. Therefore $y\in\overline{S(\mathbb{N})\cup X}$ which proves $y\in\mathbf{X}$. Q.E.D.

EXAMPLE 2.5. – There exist sequentially compact SUS-spaces which are not absolutely US-closed. By virtue of the above proposition it is enough to find sequentially compact SUS-spaces satisfying i) and ii). Such a space is for example the space of all ordinals less than the initial ordinal of cardinality $2^{\aleph_0}$ provided with the order topology. Since finite products of sequentially compact SUS-spaces satisfying i) and ii) have again the same properties we obtain immediately infinitely many such examples.

The following conjecture seems rather plausible.

CONJECTURE. – *Every sequentially compact* US-*space of cardinality less than* $2^{\aleph_0}$ *is absolutely* US-*closed.*

By 2.2 and 2.3 the class of absolutely US-closed spaces is contained in the class of all sequentially compact US-spaces and contains the class of all countable sequentially compact US-spaces. We do not know if it is completely determined by the cardinality of the underlying space (in the spirit of the above conjecture). However the following can be proved.

THEOREM 2.6. – *Let* X *be absolutely* US-*closed and let* $f\colon X\to Y$

be a continuous map onto a US*-space* Y. *Then* Y *is absolutely* US*-closed.*

PROOF. – Assume that Y is not absolutely US-closed. Then Y can be embedded into a space $Y' \in \mathrm{US}$ such that there exists $y \in \mathbf{Y}$ in Y' and $y \notin Y$. By lemma 2.2 X and consequently Y are sequentially compact. Let S be a sequence in Y' such that $y = \lim S$ and $y \in \overline{S \circ S'(\mathbb{N}) \cap Y}$ for every $S' \in \mathrm{MON}$. Clearly S is eventually in $Y' \setminus Y$ because of $Y' \in \mathrm{US}$ and the sequential compactness of Y. We can assume wlog that S is a one-to-one sequence in $Y' \setminus Y$.

By means of this sequence we construct a US-space Z containing X as a subspace. Set $Z = \{z\} \cup \mathbb{N} \cup X$ and extend f to Z defining $f(z) = y$ and $f(n) = S(n)$ for every $n \in \mathbb{N}$. Define a topology on X in the following way: the points of $\mathbb{N}$ are isolated; a basic *nbd* of a point $x \in X$ has the form $U \cup F$ where U is an open *nbd* of x in X and $F = \{n \in \mathbb{N} : S(n) \in W\}$ where W is an open *nbd* of $f(x)$ in Y' such that $f(U) \subset W$; finally the *nbds* of z are the proimages $f^{-1}(W)$ of *nbds* W of y in Y'. Denote by S_0 the sequence in Z defined by the identity of $\mathbb{N}$. The extension Z of X has the following properties:

i) a sequence S' in X converges to z in Z iff $f \circ S'$ converges to y in Y';

ii) a subsequence S'' of S_0 converges to a point $x \in X$ iff $f \circ S''$ converges to $f(x)$ in Y';

iii) for a subset D of $\mathbb{N}$ and $x \in X$, $x \in \overline{D}$ in Z iff $f(x) \in \overline{f(D)}$ in Y';

iv) for a subset D of X $z \in \overline{D}$ in Z iff $y \in \overline{f(D)}$ in Y'.

Now i), ii) and $Y' \in \mathrm{US}$ yield $Z \in \mathrm{US}$. Let S'' be a subsequence of S_0, then $f(\overline{S''(\mathbb{N}) \cap X}) = \overline{f \circ S''(\mathbb{N}) \cap Y}$ by iii). Since $y \in \mathbf{Y}$ in Y', it follows that $y \in \overline{f \circ S''(\mathbb{N}) \cap Y}$ in Y'. By iv) this gives $z \in \overline{S''(\mathbb{N}) \cap X}$. Therefore $z \in \mathbf{X}$ in Z. This contradicts the fact that X is absolutely US-closed. Q.E.D.

Finally we show that the classicall notion of $\mathfrak{T}$-closedness is too restrictive for $\mathfrak{T} = \mathrm{US}$ and SUS.

THEOREM 2.7. – *Every* US*-closed space is finite.*

PROOF. – By 1.3 every US-closed space is sequentially compact. Let now X be an infinite sequentially compact US-space, we are going to show that X is not US-closed.

Let S_0 be a one-to-one convergent sequence in X and let

$x = \lim S_0$. Let φ be a non-fixed ultrafilter on $S_0(\mathbb{N})$. We denote by φ also the ultrafilter generated by φ in X.

Denote by $\mathcal{F}$ the filter on X generated by the open sets belonging to φ. Clearly the *nbd* filter $\mathcal{U}_x$ of x is contained in $\mathcal{F}$, on the other hand $X\setminus\{x\} \in \mathcal{F}$, so $\bigcap\{U: U \in \mathcal{F}\} = \varphi$.

For every sequence S in $X_{\mathcal{F}}$ which is not eventually the constant sequence $\{\mathcal{F}\}$ and $\{\mathcal{F}\} \in \lim S$, $x \in \lim S\circ S'$ holds for every $S' \in$ MON such that the sequence $S\circ S'$ is definitely in X.

Let S be a convergent sequence in $X_{\mathcal{F}}$. If $\{\mathcal{F}\} \notin \lim S$, then clearly S has a unique limit point in $X_{\mathcal{F}}$ since $X \in$ US and X is open in X, so S is eventually in X.

Suppose that $\{\mathcal{F}\} \in \lim S$. If S has no subsequence in X, then clearly S has a unique limit point. Assume that S has an infinite subsequence in X. In such a case we can assume wlog that S is entirely in X. It was remarked already that in such a case $x \in \lim S$. We consider now two cases: 1° $S(\mathbb{N}) \cap S_0(\mathbb{N})$-finite; 2° S and S_0 have a common subsequences. In the case 1° for $A = S_0(\mathbb{N})$ observe that $[S(\mathbb{N})]_S \cap A$ is finite since S converges to x in X and $X \in$ US. By lemma 1.2 there exists $S' \in$ MON such that $\overline{S\circ S'(\mathbb{N})} \cap A$ is finite. Then $U = X\setminus\overline{S\circ S'(\mathbb{N})} \in \mathcal{F}$ and $\{\mathcal{F}\} \cup U$ is a *nbd* of $\{\mathcal{F}\}$ in $X_{\mathcal{F}}$ which avoids the subsequence $S\circ S'$ of S. This contradicts $\{\mathcal{F}\} \in \lim S$.

In the case 2° we can assume wlog that S is a subsequence of S_0. Since φ is an ultrafilter there exist $W \in \varphi$ with $W \not\ni x$ and $S' \in$ MON such that $S\circ S'$ is out of W. As noted above $S\circ S'$ converges to x, this is why $X \in$ US yields $[S\circ S'(\mathbb{N})]_S \cap W = \emptyset$. By lemma 1.2 there exists $S'' \in$ MON such that $\overline{S\circ S'\circ S''(\mathbb{N})} \cap W = \emptyset$. Then $U = X\setminus\overline{S\circ S'\circ S''(\mathbb{N})}$ is open in X and $U \supset W$, hence $U \in \varphi$ and consequently $U \in \mathcal{F}$. Therefore $\{\mathcal{F}\} \cup U$ is an open *nbd* of $\{\mathcal{F}\}$ in $X_{\mathcal{F}}$ which avoids $S\circ S'\circ S''$, so S does not converge to $\{\mathcal{F}\}$ —a contradiction. Q.E.D.

THEOREM 2.8. – *The* SUS-*closed spaces are precisely the* SUS-*spaces which are finite unions of convergent sequences. In particular every* SUS-*closed space is compact.*

PROOF. – The sufficiency is obvious.

Let X be an infinite SUS-closed spaces. We are going to show that every non-isolated point x of X has a *nbd* consisting of the range of a convergent one-to-one sequence in X (it will converge to x necessarily). Assume the contrary, then for every open *nbd* U of x and for every sequence S in X converging to x the set

$$U_S = U\setminus(S(\mathbb{N}) \cup \{x\}) \neq \emptyset \tag{3}$$

is open in X, since $S(\mathbb{N}) \cup \{x\}$ is closed because of $X \in \mathrm{SUS}$. Denote by $\mathcal{F}$ the open filter generated by the sets (3). Clearly every sequence S' in $X_{\mathcal{F}}$ for which $\{\mathcal{F}\}$ is a cluster point has also x as a cluster point. By the choice of $\mathcal{F}$ it follows that no convergent sequence in $X_{\mathcal{F}}$ can have $\{\mathcal{F}\}$ as a cluster point. Therefore $X_{\mathcal{F}} \in \mathrm{SUS}$ which contradicts the fact that X is SUS-closed since X is dense in $X_{\mathcal{F}}$.

It was proved in this way that for every $x \in X$ there exists a sequence S in X converging to x, such that $S(\mathbb{N})$ is a *nbd* of x. In particular X is a sequential space.

Assume now that X is not a finite union of convergent sequences. Consider the open filter $\mathcal{K}$ on X generated by the complements of all possible unions

$$\bigcup_{m=1}^{n} \{x_m\} \cup S_m(\mathbb{N}) \,, \tag{4}$$

where S_m is a sequence in X converging x_m. By the assumption the complements of the sets (4) are non-empty. On the other hand by Lemma 3.7.1 in [11] $X_{\mathcal{K}} \in \mathrm{SUS}$. This is a contradiction since X is SUS-closed. Q.E.D.

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