

## SEQUENTIALLY $\mathcal{P}$ - CLOSED SPACES (\*)

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**SOMMARIO.** - Un  $\mathcal{P}$  - spazio è sequenzialmente  $\mathcal{P}$  - chiuso se e solo se esso è sequenzialmente chiuso in ogni  $\mathcal{P}$  - spazio in cui esso sia immerso. Se  $\mathcal{P}$  è una classe di spazi, rispettivamente, completamente regolari, normali, perfettamente normali, localmente compatti, paracompatti, metrici, gli spazi sequenzialmente  $\mathcal{P}$  - chiusi sono esattamente i  $\mathcal{P}$  - spazi numerabilmente compatti. Per varie categorie  $\mathcal{P}$  consistenti di spazi di Hausdorff si danno caratterizzazioni interne degli spazi sequenzialmente  $\mathcal{P}$  - chiusi che permettono di stabilirne molte altre proprietà.

**SUMMARY.** - A  $\mathcal{P}$  - space is sequentially  $\mathcal{P}$  - closed if it is sequentially closed in every  $\mathcal{P}$  - space in which it is embedded. For  $\mathcal{P}$  - completely regular, normal, perfectly normal, locally compact, paracompact and metric the sequentially  $\mathcal{P}$  - closed spaces are precisely the countably compact  $\mathcal{P}$  - spaces. Internal characterization of sequentially  $\mathcal{P}$  - closed spaces are given for various categories  $\mathcal{P}$  consisting of Hausdorff spaces which permits to establish a lot of other properties of the sequentially  $\mathcal{P}$  - closed spaces.

### 0. Introduction.

A space  $Y$  is called *sequentially determined extension* of its subspace  $X$  iff for every point  $y \in Y$  there exists a sequence  $\{x_n\}_{n=1}^{\infty}$  in  $X$  such that  $\lim_{n \rightarrow \infty} x_n = y$  [Go]. Let  $\mathcal{P}$  be a class of topological spaces. A space  $X \in \mathcal{P}$  is said to be  $\mathcal{P}$  - closed (sequentially  $\mathcal{P}$  - closed) iff  $X$  is closed (sequentially closed) in every  $\mathcal{P}$  space in which it is embedded. In other words  $X$  is sequentially  $\mathcal{P}$  - closed iff  $X$  has no sequentially determined extension  $Y \in \mathcal{P}$  and  $Y \neq X$  (this holds under very mild restrictions on  $\mathcal{P}$  which are verified in all cases considered here).

Obviously, every  $\mathcal{P}$  - closed space is sequentially  $\mathcal{P}$  - closed. The

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$\mathfrak{A}$ -closed spaces were extensively studied for categories  $\mathfrak{A}$  consisting of Hausdorff spaces (see [BPS]). For such  $\mathfrak{A}$  every compact (countably compact)  $\mathfrak{A}$ -space is  $\mathfrak{A}$ -closed (sequentially  $\mathfrak{A}$ -closed). The sequentially Hausdorff-closed spaces were introduced by P. Alexandroff and P. Urysohn [AU]. They were proved that the regular sequentially Hausdorff-closed spaces coincide with the regular countably compact spaces. In [DGo] sequentially  $\mathfrak{A}$ -closed spaces for some classes  $\mathfrak{A}$  of topological spaces between  $T_1$  and  $T_2$  are studied. It is proved in particular that for the category  $SUS$  of topological spaces in which every convergent sequence has a unique accumulation point the sequentially  $SUS$ -closed spaces are precisely the countably compact  $SUS$ -spaces. On the other hand it follows by results of A. Tozzi [To] that the sequentially  $SUS$ -closed spaces coincide with the absolutely  $SUS$ -closed spaces, introduced by D. Dikranjan and E. Giuli [DG3] in a more general situation. This was in fact the starting point of our study of sequentially  $\mathfrak{A}$ -closed spaces.

The aim of this paper is to study the sequentially  $\mathfrak{A}$ -closed spaces for  $\mathfrak{A} = S(n)$ , regular, completely Hausdorff, completely regular, normal, perfectly normal, locally compact, paracompact and metric.

In section 1 we introduce open elementary filters which serve as the main tool in the study of sequentially  $\mathfrak{A}$ -closed spaces. We introduce also  $\theta^n$ -convergence and  $S^n$ -convergence (generalizations of the  $\theta$ -convergence introduced by Veličko [Ve] and the usual convergence) and characterize the  $S(n)$ -spaces by means of these convergences following Veličko [Ve], Dikranjan and Giuli [Di], [DG1], [DG3].

In section 2 we give internal characterization of the sequentially  $\mathfrak{A}$ -closed spaces in terms of special filters and covers for  $\mathfrak{A} = S(n)$ , regular, completely Hausdorff and completely regular. For  $\mathfrak{A} = S(n)$  this characterization involves the  $\theta^n$ -convergence and  $S^n$ -convergence. An example of a sequentially Hausdorff-closed completely Hausdorff space (and hence  $S(n)$ -space for  $n = 1, 2, \dots$ ) is given which is neither Hausdorff-closed nor countably compact. We give also an example of a sequentially regular-closed space which is neither regular-closed nor countably compact. We discuss also the relations between sequentially  $\mathfrak{A}$ -closed spaces and  $\mathfrak{A}(1)$ -closed spaces (see [BPS]).

In section 3 various properties of the sequentially  $\mathfrak{A}$ -closed spaces are established. We show in particular that for  $\mathfrak{A}$ -completely regular, normal, perfectly normal, locally compact, paracompact and metric the sequentially  $\mathfrak{A}$ -closed spaces are precisely the countably compact  $\mathfrak{A}$ -spaces. Since for paracompact spaces countable compactness coincides with compactness in the last two cases we get the compact  $\mathfrak{A}$ -spaces.

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### 1. Preliminaries.

Troughout the paper the properties of regularity, completely regularity, etc. include the  $T_1$  separation property,  $\bar{U}$  denotes the closure of the set  $U$  in a given topological space,  $N$  denotes the set of positive integers and  $R$  denotes the real line with the usual topology. In general the terminology and notation follow [En].

Let  $X$  be a topological space and let  $A$  be a countable subset of  $X$ . A *maximal open elementary filter generated by  $A$*  is the open filter on  $X$  with base  $\mathcal{B} = \{F \mid F \subset X, F \text{ is open in } X \text{ and } A \setminus F \text{ is finite}\}$ . An open filter  $\mathcal{F}$  on  $X$  is a *maximal open elementary filter*, iff there exists a countable subset  $A$  of  $X$ , such that  $\mathcal{F}$  is the maximal open elementary filter on  $X$  generated by  $A$ . If  $X$  is a  $US$ -space (every convergent sequence in  $X$  has a unique limit point [MN]) then  $A$  is uniquely determined by  $\mathcal{F}$  up to a finite subset (see [DGo]). An open filter will be called *open elementary filter*, iff it is contained in some maximal open elementary filter. A filter  $\mathcal{F}$  is *free* iff  $\bigcap \{F \mid F \in \mathcal{F}\} = \emptyset$ .

Let  $\mathcal{F} = \{F_\alpha\}_{\alpha \in \mathcal{A}}$  be a free open elementary filter on  $X$ . On the set  $X_{\mathcal{F}} = X \cup \{\mathcal{F}\}$  we introduce the following topology: The set  $X$  is open, the relative topology of  $X$  coincides with the original topology of  $X$  and for open base at the point  $\{\mathcal{F}\}$  we take the family  $\{\{\mathcal{F}\} \cup F_\alpha \mid F_\alpha \in \mathcal{F}, \alpha \in \mathcal{A}\}$ . It is easy to verify that the topological space obtained in this way is a  $T_1$  sequentially determined extension of  $X$  whenever  $X$  is  $T_1$ . In the sequel  $X_{\mathcal{F}}$  will be called *standard sequentially determined extension of  $X$  by  $\mathcal{F}$* .

Let  $X$  be a topological space,  $M \subset X$  and  $n \in N$ . The point  $x \in X$  is  *$S(n)$ -separated from  $M$*  iff there exist open sets  $U_i, i = 1, 2, \dots, n$  such that  $x \in U_1 \subset \dots \subset U_n, \bar{U}_i \subset U_{i+1}$  and  $\bar{U}_n \cap M = \emptyset$ ;  $x$  is  *$S(0)$ -separated from  $M$*  iff  $x \notin \bar{M}$ . The space  $X$  is an  *$S(n)$ -space*, iff every two different points in  $X$  are  $S(n)$  separated [Vi]. It is obvious that  $S(1) = T_2, S(2) = T_{2,5}$  (distinct points can be separated by disjoint closed neighbourhoods).

An open cover  $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$  of the space  $X$  is an  *$S(n)$ -cover*, iff every point  $x \in X$  is  $S(n)$ -separated from some  $X \setminus U_\alpha$ . Clearly the  $S(0)$ -covers are exactly the open covers. A filter  $\mathcal{F}$  on  $X$  is an  *$S(n)$ -filter*, iff every point  $x \in X$ , which is not adherent point for  $\mathcal{F}$  is  $S(n)$ -separated from some  $U \in \mathcal{F}$  [PV].

Let  $X$  be a topological space and  $n \in N$ . The point  $x \in X$  will be called  *$S^n$ -limit ( $\theta^n$ -limit)* of a sequence  $\{x_n\}_{n=1}^\infty$  in  $X$ , iff for every chain  $U_1 \subset U_2 \subset \dots \subset U_n$  of open neighbourhoods of  $x$  such that  $\bar{U}_i \subset U_{i+1}$  for

$i = 1, 2, \dots, n - 1$ ,  $U_n(\bar{U}_n)$  contains all but a finite number of the members of the sequence. Every sequence which has an  $S^n$ -limit ( $\theta^n$ -limit) will be called  $S^n$ -convergent ( $\theta^n$ -convergent). If for every chain  $U_1 \subset U_2 \subset \dots \subset U_n$  of open neighbourhoods of  $x$  such that  $\bar{U}_i \subset U_{i+1}$  for  $i = 1, 2, \dots, n - 1$ ,  $U_n(\bar{U}_n)$  contains infinitely many members of the sequence then  $x$  will be called  $S^n$ -adherent point ( $\theta^n$ -adherent point) of the sequence  $\{x_n\}_{n=1}^\infty$ . Clearly, the  $S^1$ -convergence is the usual topological convergence and the  $\theta^1$ -convergence is the  $\theta$ -convergence defined by Veličko [Ve]. Also every  $S^n$ -convergent sequence is  $\theta^n$ -convergent and every  $\theta^n$ -convergent sequence is  $S^{n+1}$ -convergent. So every convergent, in the usual sense, sequence is  $S^n$ -convergent and  $\theta^n$ -convergent for every  $n \in \mathbb{N}$ . If  $X$  is regular then the  $S^n$ -convergence and the  $\theta^n$ -convergence coincides with the usual convergence. Similar facts are valid for  $S^n$  and  $\theta^n$ -adherent points of a sequence. In the same way  $S^n$  ( $\theta^n$ ) convergence and  $S^n$  ( $\theta^n$ ) adherent points can be defined for nets. (For  $\theta^n$ -adherent points see [DG3] and for  $n = 1$  see also [DG2]).

The  $S(n)$ -spaces can be characterized by means of the  $\theta^n$ -convergence,  $S^n$ -convergence and the usual convergence.

**PROPOSITION 1.1.** *Let  $X$  be a topological space and  $n \in \mathbb{N}$ . The following conditions are equivalent:*

- (a)  $X$  is an  $S(n)$ -space
- (b) every convergent sequence in  $X$  has a unique  $\theta^n$ -adherent point
- (c) every convergent sequence in  $X$  has a unique  $\theta^n$ -limit
- (d) every convergent sequence in  $X$  has a unique  $S^{n+1}$ -adherent point
- (e) every convergent sequence in  $X$  has a unique  $S^{n+1}$ -limit.

*Proof.* Follows directly from the definition.

## 2. Characterization of the sequentially $\varpi$ -closed spaces.

The next theorem characterizes sequentially  $S(n)$ -closed spaces. Analogous results for  $S(n)$ -closed spaces were obtained in [PV] and for  $S(n)$ - $\theta$ -closed spaces were obtained in [DG3].

**THEOREM 2.1.** *Let  $X$  be a  $T_1$  space and  $n \in \mathbb{N}$ . The following conditions are equivalent:*

- (a) every sequence in  $X$  has a  $\theta^n$ -adherent point

- (b) every sequence in  $X$  has an  $S^{n+1}$ -adherent point
  - (c) every countable  $S(n)$ -cover of  $X$  has a finite subcover
  - (d) every  $S(n)$ -filter with a countable base of closed sets has an adherent point
  - (e) every open elementary  $S(n)$ -filter has an adherent point
  - (f) every maximal open elementary  $S(n)$ -filter has an adherent point.
- If  $X$  is an  $S(n)$ -space then the above conditions are equivalent to:
- (g)  $X$  is sequentially  $S(n)$ -closed.

*Proof.* Obviously (a) implies (b) and (c) implies (f). To see that (b) implies (c) assume that  $\{U_n\}_{n=1}^{\infty}$  is a countable  $S(n)$ -cover which has no finite subcover. For every  $k = 1, 2, \dots$  we choose  $x_k \notin \bigcup_{i=1}^k U_i$ . Let  $x \in X$ . Since  $\{U_i\}_{i=1}^{\infty}$  is an  $S(n)$ -cover then there exists an element  $U_i$  of the cover and chain  $V_1 \subset V_2 \dots \subset V_n$  of open neighbourhoods of  $x$  such that  $\overline{V_j} \subset V_{j+1}$  for  $j = 1, 2, \dots, n-1$  and  $\overline{V_n} \subset U_i$ . This means that  $x$  is not an  $S^{n+1}$ -adherent point for  $\{x_k\}_{k=1}^{\infty}$  since for every  $i \in N$  and  $k \geq i$  we have  $x_k \notin U_i$ . Therefore  $\{x_k\}_{k=1}^{\infty}$  has no  $S^{n+1}$ -adherent points in  $X$ : a contradiction. Let now  $\mathcal{F}$  be an  $S(n)$ -filter on  $X$  with countable closed base  $\{F_i\}_{i=1}^{\infty}$ . Assume that  $\mathcal{F}$  has no adherent points. Then  $\mathcal{U} = \{U_i \mid U_i = X \setminus F_i, i \in N\}$  is an  $S(n)$ -cover of  $X$ . Let  $U_{i_1}, U_{i_2}, \dots, U_{i_k}$  be a finite subcover of  $\mathcal{U}$ . Since  $\mathcal{F}$  is a filter we have  $\bigcap_{j=1}^k (X \setminus U_{i_j}) = \bigcap_{j=1}^k F_{i_j} \neq \phi$ , hence  $U_{i_1}, U_{i_2}, \dots, U_{i_k}$  is not a cover of  $X$ . This contradiction proves that (c) implies (d). Now we will prove that (d) implies (e). Let  $\mathcal{F}$  be an open elementary  $S(n)$ -filter without adherent points. There exists a maximal open elementary filter  $\mathcal{F}'$  such that  $\mathcal{F} \subset \mathcal{F}'$ . Then  $\mathcal{F}'$  has no adherent points and if  $\{x_k\}_{k=1}^{\infty}$  determines  $\mathcal{F}'$  then  $\{x_k\}_{k=i}^{\infty}$  is a closed set for every  $i \in N$ . Thus the filter  $\mathcal{F}''$  generated by  $\{F_i \mid F_i = \{x_k\}_{k=i}^{\infty}, i \in N\}$  contains  $\mathcal{F}'$ . So  $\mathcal{F}''$  is an  $S(n)$ -filter with a countable base of closed sets without adherent points. To see that (f) implies (a) assume that  $\{x_k\}_{k=1}^{\infty}$  is a sequence in  $X$  which has no  $\theta^n$ -adherent points. Let  $\mathcal{F}$  be the maximal open elementary filter generated by  $\{x_k\}_{k=1}^{\infty}$  and  $x$  be an arbitrary point

of  $X$ . There exists a chain  $U_1 \subset U_2 \subset \dots \subset U_n$  of open neighbourhoods of  $x$  such that  $\overline{U_i} \subset U_{i+1}$ ,  $i = 1, 2, \dots, n-1$  and  $X \setminus \overline{U_n}$  contains all but a finite members of the sequence. Therefore  $X \setminus \overline{U_n} \in \mathcal{F}$ . On the other hand  $x$  and  $X \setminus \overline{U_n}$  are  $S(n)$ -separated. This means that  $\mathcal{F}$  is a maximal open elementary  $S(n)$ -filter with no adherent points. Contradiction. Now let we assume that  $X$  is not sequentially  $S(n)$ -closed. Thus there exists an  $S(n)$ -space  $Y \supset X$ , a point  $y \in Y \setminus X$  and a sequence  $\{x_k\}_{k=1}^{\infty}$  in  $X$  such that  $\lim_{k \rightarrow \infty} x_k = y$ . Clearly  $y$  is a  $\theta^n$ -adherent point of  $\{x_k\}_{k=1}^{\infty}$ . From Proposition 1.1 it follows that  $\{x_k\}_{k=1}^{\infty}$  has no other  $\theta^n$ -adherent points in  $Y$ . Thus  $\{x_k\}_{k=1}^{\infty}$  has no  $\theta^n$ -adherent points in  $X$ . This contradiction proves that (a) implies (g). To prove that (g) implies (f) assume that  $\mathcal{F}$  is a maximal open elementary  $S(n)$ -filter with no adherent points. Let  $X_{\mathcal{F}}$  be the standard sequentially determined extension of  $X$  by  $\mathcal{F}$ . It is easy to verify that  $X_{\mathcal{F}}$  is an  $S(n)$ -space. Thus  $X$  is not sequentially  $S(n)$ -closed. Contradiction.

The idea to characterize closed spaces with  $\theta$ -convergence and elementary filters (but in a somewhat different sense, see [Bo], chap. 1, § 6) comes from Veličko [Ve].

Now we show that the class of sequentially  $S(n)$ -closed spaces is not exhausted by the  $S(n)$ -closed spaces and by the countably compact  $S(n)$ -spaces.

**EXAMPLE 2.2.** Let  $N$  be the space of positive integers with the discrete topology and let  $\beta N$  be the Čech-Stone compactification of  $N$ . Let also  $X = (\beta N \setminus W) \cup \{x_{ij}\}_{i,j=1}^{\infty} \cup \{y_i\}_{i=1}^{\infty}$ . We provide  $X$  with a topology as follows: The points  $\{x_{ij}\}_{i,j=1}^{\infty}$  are isolated for  $i \in N$  and  $j \in N$ . For a neighbourhood base of  $y_i$  ( $i \in N$ ) we take the family  $y_i \cup \{\{x_{ij}\}_{j=k}^{\infty}, k \in N\}$ . Let  $\{\mathcal{F}\} \in \beta N \setminus W$  and let  $\{U_{\alpha}, \alpha \in \mathcal{A}\}$  be a neighbourhood base of  $\{\mathcal{F}\}$  in  $\beta N$ . For a neighbourhood base of  $\{\mathcal{F}\}$  in  $X$  we take the family  $\{V_{\alpha} \mid V_{\alpha} = (U_{\alpha} \setminus W) \cup \{\{x_{ij}\}_{i,j=1}^{\infty}, i \in U_{\alpha} \cap N\}, \alpha \in \mathcal{A}\}$ . It is easy to verify that  $X$  is a Hausdorff-closed  $S(n)$ -space for every  $n \in N$  and  $X$  is not countably compact. Let now  $\mathcal{F} \in \beta N \setminus W$  and  $Y = X \setminus \{\mathcal{F}\}$ . Clearly  $Y$  is  $S(n)$ -closed for no  $n$  and  $Y$  is not countably compact. By (a) of the above theorem  $Y$  is sequentially  $S(n)$ -closed.

Let  $X$  be a topological space. An open filter  $\mathcal{F}$  on  $X$  is a *regular filter* iff for each  $U \in \mathcal{F}$  there exists  $V \in \mathcal{F}$  such that  $\overline{V} \subset U$  [Ba]. Let  $\mathcal{U}$  and

$\mathcal{V}$  be open covers of a space  $X$ .  $\mathcal{V}$  is a *shrinkable refinement* of  $\mathcal{U}$  iff for each  $V \in \mathcal{V}$ , there is  $U \in \mathcal{U}$  such that  $\bar{V} \subset U$ . An open cover  $\mathcal{U}$  is *regular* iff there exists an open cover  $\mathcal{V}$  which refines  $\mathcal{U}$  and  $\mathcal{V}$  is a shrinkable refinement of itself [BPS].

**THEOREM 2.3.** *Let  $X$  be a  $T_1$  space. The following conditions are equivalent:*

- (a) *every open elementary regular filter on  $X$  has adherent points*
- (b) *every countable regular cover of  $X$  has a finite subcover.*  
*If  $X$  is a regular space then the above conditions are equivalent to:*
- (c)  *$X$  is sequentially regular-closed.*

*Proof.* To see that (a) implies (b) assume that  $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$  is a countable regular cover without finite subcovers. For each  $k = 1, 2, \dots$  we choose a point  $x_k \in X$  such that  $x_k \notin \bigcup_{i=1}^k U_i$ . Then clearly  $x_k \notin U_i$  whenever  $k \geq i$ . Let  $\mathcal{V} = \{V_{\alpha}\}_{\alpha \in \mathcal{A}}$  be a cover of  $X$  which refines  $\mathcal{U}$  and  $\mathcal{V}$  is a shrinkable refinement of itself. Clearly the cover  $\mathcal{V}$  has no finite subcovers. It is easy to verify that the filter  $\mathcal{F}$  generated by the filter base  $\left\{X \setminus \bigcup_{i=1}^k V_{\alpha_i}, \alpha_i \in \mathcal{A}, k \in \mathbb{N}\right\}$  is an open elementary regular filter on  $X$  without adherent points. Contradiction. Let now  $\mathcal{F}$  be an open elementary regular filter on  $X$  without adherent points. There exists a maximal open elementary filter such that  $\mathcal{F} \subset \mathcal{F}'$ . Let  $\{x_k\}_{k=1}^{\infty}$  determines  $\mathcal{F}'$ . But  $\mathcal{F}'$  has no adherent points. Then  $\{x_k\}_{k=1}^{\infty}$  is a closed set for every  $i \in \mathbb{N}$ . Let  $\mathcal{U} = \{U_i \mid U_i = X \setminus \{x_k\}_{k=i}^{\infty}, i \in \mathbb{N}\}$  and  $\mathcal{V} = \{V \mid \text{there exists an open set } W \in \mathcal{F} \text{ such that } V = X \setminus \bar{W}\}$ . It is easy to verify that  $\mathcal{V}$  is an open cover of  $X$  which refines  $\mathcal{U}$  and  $\mathcal{V}$  is a shrinkable refinement of itself. Thus  $\mathcal{U}$  is a countable regular cover of  $X$  without finite subcovers and this proves that (b) implies (a). Now we prove that (a) implies (c). Assume that  $X$  is not sequentially regular - closed. Thus there exists a regular space  $Y \supset X$ , a point  $y \in Y \setminus X$  and a sequence  $\{x_n\}_{n=1}^{\infty}$  in  $X$  such that  $\lim_{n \rightarrow \infty} x_n = y$ . Let  $\mathcal{V}_y$  be the filter of neighbourhoods of  $y$  on  $Y$ .

Since  $Y$  is a regular space it follows that  $\mathcal{V}_y$  is an open elementary regular

filter on  $Y$  with no adherent points in  $X$ . Then  $\mathcal{V} = \{V \mid \text{there exists } W \in \mathcal{V}_y \text{ such that } V = X \cap W\}$  is an open elementary regular filter on  $X$  without adherent points. Contradiction. Assume that there exists an open elementary regular filter  $\mathcal{F}$  on  $X$  without adherent points. Then the standard sequentially determined extension  $X_{\mathcal{F}}$  of  $X$  by  $\mathcal{F}$  will be a regular space. This contradicts the sequentially regular - closedness of  $X$ , so (c) implies (a).

Now we show that the class of sequentially regular - closed spaces is not exhausted by the regular - closed spaces and by the countably compact regular spaces.

EXAMPLE 2.4. The space  $X$  in Example 4.18 in [BPS] is a minimal regular space which is not countably compact [BS]. Let  $x = (\omega_1, 1, 1) = (\omega_1, 1, 2)$  and  $Y = X \setminus \{x\}$ . Then by Lemma 3.10  $Y$  is a sequentially regular - closed space which is neither regular - closed nor countably compact.

Let  $X$  be a topological space.  $X$  is *completely Hausdorff* iff for each pair  $x, y$  of distinct points, there exists a continuous real - valued function  $f$  such that  $f(x) \neq f(y)$ . An open filter  $\mathcal{F}$  on  $X$  is *completely Hausdorff* iff for each  $x \in X$  which is not an adherence point of  $\mathcal{F}$  there exists an open set  $U$  containing  $x$ ,  $V \in \mathcal{F}$  and continuous real - valued function  $f$  on  $X$  such that  $f(U) = \{1\}$  and  $f(V) = \{0\}$ . An open filter  $\mathcal{F}$  on  $X$  is *completely regular* iff for each  $U \in \mathcal{F}$ , there exists  $V \in \mathcal{F}$  and a continuous real - valued function  $f$  on  $X$  such that  $f(V) = \{0\}$  and  $f(X \setminus U) = \{1\}$ . Let  $\mathcal{V}$  and  $\mathcal{U}$  be covers of a space  $X$ .  $\mathcal{V}$  is a *continuous refinement* of  $\mathcal{U}$  iff for each  $V \in \mathcal{V}$  there is  $U \in \mathcal{U}$  and continuous real - valued function  $f$  on  $X$  such that  $f(V) = \{0\}$  and  $f(X \setminus U) = \{1\}$ . An open cover is *completely Hausdorff* iff it has a continuous refinement [BPS]. An open cover  $\mathcal{U}$  is *completely regular* iff there is an open cover  $\mathcal{V}$  which refines  $\mathcal{U}$  and  $\mathcal{V}$  is a continuous refinement of itself.

THEOREM 2.5. *Let  $X$  be a  $T_1$  space. The following conditions are equivalent:*

- (a) *every countable completely Hausdorff cover of  $X$  has a finite sub-cover*
- (b) *every open elementary completely Hausdorff filter on  $X$  has adherent points*
- (c) *every maximal open elementary completely Hausdorff filter on  $X$  has adherent points.*

*If  $X$  is a completely Hausdorff space then the above conditions are equivalent to:*

(d)  $X$  is sequentially completely Hausdorff - closed.

*Proof.* To see that (a) implies (b) let  $\mathcal{F}$  be an open elementary completely Hausdorff filter on  $X$  without adherent points. There exists a maximal open elementary filter  $\mathcal{F}'$  on  $X$  such that  $\mathcal{F} \subset \mathcal{F}'$ . If  $\{x_k\}_{k=1}^{\infty}$  is a sequence which generate  $\mathcal{F}'$  then  $\{x_k\}_{k=1}^{\infty}$  is a closed set for every  $i \in \mathbb{N}$ . Let  $U = \{U_i \mid U_i = X \setminus \{x_k\}_{k=i}^{\infty}, i \in \mathbb{N}\}$  and  $\mathcal{V} = \{V_x \mid V_x = f^{-1}[0, \nu_2), x \in X, W \in \mathcal{F} \text{ and } f(W) = \{1\}\}$ . Then  $\mathcal{U}$  and  $\mathcal{V}$  are open covers of  $X$  and  $\mathcal{V}$  is a continuous refinement of  $\mathcal{U}$ . Thus  $\mathcal{U}$  is a completely Hausdorff cover of  $X$  without a finite subcover. Obviously (b) implies (c). We prove that (c) implies (a). Let  $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$  be a countable completely Hausdorff cover of  $X$  without finite subcovers. Let  $\mathcal{V} = \{V_{\alpha} \mid \alpha \in \mathcal{A}\}$  be a cover of  $X$  which is a continuous refinement of  $\mathcal{U}$ . For every  $k \in \mathbb{N}$  we choose  $x_k \notin \bigcup_{i=1}^k U_i$  and let  $\mathcal{F}$  be the maximal open elementary filter generated by  $\{x_k\}_{k=1}^{\infty}$ . If  $x$  is an arbitrary point of  $X$  then there exists  $\alpha \in \mathcal{A}$  such that  $x \in V_{\alpha}$  and there exists  $i \in \mathbb{N}$  and  $f : X \rightarrow \mathcal{R}$  such that  $f(V_{\alpha}) = \{0\}$  and  $f(X \setminus U_i) = \{1\}$ . Let  $W = f^{-1}(\nu_2, 1]$  and  $g(x) = 2 \cdot \min(f(x), \nu_2)$ . Then  $W \in \mathcal{F}$ ,  $g(V_{\alpha}) = \{0\}$  and  $g(W) = \{1\}$ . Thus  $\mathcal{F}$  is a maximal open elementary filter which is completely Hausdorff and it has no adherent points. To prove that (b) implies (d) assume that  $X$  is not sequentially completely Hausdorff - closed. Then there exists a completely Hausdorff space  $Y \supset X$ , a point  $y \in Y \setminus X$  and a sequence  $\{x_k\}_{k=1}^{\infty}$  of points of  $X$  such that  $\lim_{k \rightarrow \infty} x_k = y$ . The filter  $\mathcal{V}_y$  of neighbourhoods of  $y$  is a completely Hausdorff filter on  $Y$ . Let  $\mathcal{V} = \{V \mid \text{there exists } W \in \mathcal{V}_y \text{ such that } V = X \cap W\}$ . Then  $\mathcal{V}$  is an open elementary completely Hausdorff filter on  $X$  without adherent points. Contradiction. If there exists an open elementary completely Hausdorff filter  $\mathcal{F}$  on  $X$  without adherent points, then  $X_{\mathcal{F}}$  will be a completely Hausdorff, sequentially determined extension of  $X$ . This proves that (d) implies (b).

The space  $Y$  in Example 2.2 is also completely Hausdorff, consequently sequentially completely Hausdorff - closed. On the other hand it is neither completely Hausdorff - closed nor countably compact.

LEMMA 2.6. *In a completely regular space  $X$  every open cover of  $X$  is a completely regular cover.*

*Proof.* Let  $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$  be an open cover of  $X$ . For every  $\alpha \in \mathcal{A}$  and every  $x \in U_\alpha$  let  $f_{\alpha,x}$  be a continuous real-valued function such that  $f_{\alpha,x}(x) = 1$  and  $f_{\alpha,x}(X \setminus U_\alpha) = \{0\}$ . If  $\mathcal{V} = \{V \mid V = f_{\alpha,x}^{-1}(\frac{1}{n}, 1], \alpha \in \mathcal{A}, x \in U_\alpha, n \geq 2\}$  then  $\mathcal{V}$  is an open cover of  $X$ ,  $\mathcal{V}$  refines  $\mathcal{U}$  and  $\mathcal{V}$  is a continuous refinement of itself.

THEOREM 2.7. *Let  $X$  be a  $T_1$  space. The following conditions are equivalent:*

- (a) *every countable completely regular cover of  $X$  has a finite subcover*
  - (b) *every open elementary completely regular filter on  $X$  has adherent points.*
- If  $X$  is a completely regular space then the above conditions are equivalent to:*
- (c)  *$X$  is sequentially completely regular - closed*
  - (d)  *$X$  is countably compact.*

*Proof.* Let  $\mathcal{F}$  be an open elementary completely regular filter on  $X$  without adherent points and let  $\mathcal{F}'$  be a maximal open elementary filter such that  $\mathcal{F} \subset \mathcal{F}'$ . If  $\{x_k\}_{k=1}^\infty$  is a sequence which generates  $\mathcal{F}'$ , then  $\{x_k\}_{k=i}^\infty$  is a closed set for every  $i \in \mathbb{N}$ . Let  $\mathcal{U} = \{U_i \mid U_i = X \setminus \{x_k\}_{k=i}^\infty, i \in \mathbb{N}\}$  and  $\mathcal{V} = \{V \mid \text{there exists an open set } W \in \mathcal{F} \text{ such that } V = X \setminus \overline{W}\}$ . Then  $\mathcal{U}$  and  $\mathcal{V}$  are open covers of  $X$ ,  $\mathcal{V}$  refines  $\mathcal{U}$  and  $\mathcal{V}$  is continuous refinement of itself. Thus  $\mathcal{U}$  is a completely regular cover of  $X$  without finite subcovers. This proves that (a) implies (b). To see that (b) implies (a) let  $\mathcal{U} = \{U_i\}_{i=1}^\infty$  be a countable completely regular cover of  $X$  without finite subcovers and let  $\mathcal{V} = \{V_\alpha \mid \alpha \in \mathcal{A}\}$  be an open cover of  $X$  which refines  $\mathcal{U}$  and  $\mathcal{V}$  is a continuous refinement of itself. For every  $k \in \mathbb{N}$  we choose  $x_k \notin \bigcup_{i=1}^k U_i$  and let  $\mathcal{B} = \{W \mid \text{there exist } V_\alpha \in \mathcal{V} \text{ such that } W = X \setminus \bigcup_{i=1}^k \overline{V_\alpha}, k \in \mathbb{N}\}$ . Let  $\mathcal{F}'$  be the maximal open elementary filter on  $X$  determined by  $\{x_k\}_{k=1}^\infty$ . If  $\mathcal{F}$  is the open filter

with base  $\mathcal{B}$  then  $\mathcal{F} \subset \mathcal{F}'$  and hence  $\mathcal{F}$  is an open elementary completely regular filter without adherent points. Now we prove that (b) implies (c). Assume that  $X$  is not sequentially completely regular - closed. Then there exists a completely regular space  $Y \supset X$ , a point  $y \in Y \setminus X$  and a sequence  $\{x_k\}_{k=1}^{\infty}$  of points of  $X$  such that  $\lim_{k \rightarrow \infty} x_k = y$ . The filter  $\mathcal{V}_y$  of neighbourhoods of  $y$  is a completely regular filter on  $Y$ . Let  $\mathcal{V} = \{W\}$  there exists an open set  $V \in \mathcal{V}_y$  such that  $W = X \cap V$ , then  $\mathcal{V}$  is an open elementary completely regular filter on  $X$  without adherent points. Contradiction. If there exists an open elementary completely regular filter  $\mathcal{F}$  on  $X$  without adherent points, then  $X_{\mathcal{F}}$  will be a completely regular, sequentially determined extension of  $X$ . This proves that (c) implies (b). The equivalence of conditions (a) and (d) follows directly by Lemma 2.6.

For a class  $\varphi$ , the class of all first countable  $\varphi$  - spaces will be denoted by  $\varphi(1)$  [BPS]. Evidently every sequentially  $\varphi$  - closed  $\varphi(1)$  space is  $\varphi(1)$  - closed. Hence the sequentially  $\varphi(1)$  - closed spaces coincide with the  $\varphi(1)$  - closed spaces. For various classes  $\varphi$  the  $\varphi(1)$  - closed spaces were studied in [Ste2].

A family of open sets  $\mathcal{U}$  in a space  $X$  is a *proximate cover* of  $X$  iff  $\bigcup \{\bar{U} \mid U \in \mathcal{U}\} = X$  [Ka].

**THEOREM 2.8.** *Let  $X$  be a  $T_1$  space and  $n \in \mathbb{N}$ . The following conditions are equivalent:*

- (a) *every countable  $S(n-1)$  - cover of  $X$  contains a finite proximate subcover*
- (b) *every countable open  $S(n)$  - filter has adherent points.*  
*If  $X$  is an  $S(n)(1)$  - space then the above conditions are equivalent to:*
- (c)  *$X$  is  $S(n)(1)$  - closed.*

*Proof.* To see that (a) implies (b) suppose that  $\mathcal{F}$  is a countable open  $S(n)$  - filter on  $X$  without adherent points. Then  $\mathcal{U} = \{U \mid U = X \setminus \bar{V}, V \in \mathcal{F}\}$  is a countable  $S(n-1)$  - cover of  $X$  and  $\mathcal{U}$  has a proximate subcover.

So that if  $X = \bigcup_{i=1}^k \bar{U}_i$  then  $\bigcap_{i=1}^k (X \setminus \bar{U}_i) = \bigcap_{i=1}^k V_i = \phi$ . But  $V_i \in \mathcal{F}$  for  $i = 1, 2, \dots, k$ . Contradiction. Now let us assume that  $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$  is a countable  $S(n-1)$  - cover of  $X$  which has no finite proximate subcovers. For every

$i \in N$  we consider  $V_i = X \setminus (\bigcup_{j=1}^i \overline{U_j})$ . Obviously  $\mathcal{B} = \{V_i\}_{i=1}^{\infty}$  is a countable

open base of a filter  $\mathcal{F}$ . One can easily verify that  $\mathcal{F}$  is a countable open  $S(n)$ -filter without adherent points. This contradiction proves that (b) implies (a). If  $X$  is not  $S(n)$ (1)-closed then there exists an  $S(n)$ (1) extension  $Y$  of  $X$  and a point  $y \in Y \setminus X$ . But the trace on  $X$  of the neighbourhood filter of the point  $y$  is a countable open  $S(n)$ -filter on  $X$  without adherent points in  $X$  and this proves that (b) implies (c). To see that (c) implies (b) we suppose that  $\mathcal{F}$  is a countable open  $S(n)$ -filter without adherent points. Then the standard extension  $X_{\mathcal{F}}$  of  $X$  by  $\mathcal{F}$  is an  $S(n)$ (1) space. Contradiction.

The above theorem for  $n = 1, 2$  is proved by R. Stephenson [Ste2].

### 3. Properties of the sequentially $\varpi$ -closed spaces.

It was proved by P. Alexandroff and P. Urysohn [AU] that the regular Hausdorff - closed spaces (regular sequentially Hausdorff - closed spaces) are precisely the compact (regular countably compact) spaces. In fact every regular  $S(n)$ -closed space is compact as shown by Herlich [He] for  $n = 2$  and by Porter and Votaw [PV] for  $n > 2$ . On the other hand every completely regular, regular - closed space is compact ([He], [BS]). We show next that similar results are valid for sequentially  $\varpi$ -closed spaces.

**COROLLARY 3.1.** (a) *Let  $X$  be a regular space and  $n \in N$ . Then  $X$  is sequentially  $S(n)$ -closed iff  $X$  is countably compact.*

(b) *Let  $X$  be a completely regular space. Then  $X$  is sequentially regular - closed iff  $X$  is countably compact.*

*Proof.* (a). Follows by the fact that every open cover of a regular space is an  $S(n)$ -cover and by Theorem 2.1. (b). Follows by Theorem 2.3. and Theorem 2.7.

**COROLLARY 3.2.** *Let  $X$  be a Lindelöf, regular space and  $n \in N$ . The following conditions are equivalent:*

- (a)  *$X$  is compact*
- (b)  *$X$  is regular - closed*
- (c)  *$X$  is sequentially regular - closed*
- (d)  *$X$  is sequentially  $S(n)$ -closed.*

*Proof.* For the equivalence of (a) and (b) see [He]. The equivalence of the other conditions follows by the fact that every Lindelöf, regular space is normal [En] and by Corollary 3.1.

**THEOREM 3.3.** *Let  $X$  be a normal space. Then  $X$  is sequentially normal - closed iff  $X$  is countably compact.*

*Proof.* The proof follows immediately from Corollary 3.1 and Lemma 3.4.

**LEMMA 3.4.** *Let  $X$  be a regular space,  $x \in X$  and let  $X \setminus \{x\}$  be a normal space. Then  $X$  is a normal space.*

**THEOREM 3.5.** *Let  $X$  be a perfectly normal space. The following conditions are equivalent:*

- (a)  $X$  is perfectly normal - closed
- (b)  $X$  is sequentially perfectly normal - closed
- (c)  $X$  is countably compact.

*Proof.* For the equivalence of conditions (a) and (b) see [Ste2]. Obviously (c) implies (b). It is known that in a normal space the countable compactness coincides with the feeble compactness (see [Ste1] and [Hew]) and that a regular space  $X$  is feebly compact iff every countable open regular filter on  $X$  has adherent points [Ste1]. Then the proof that (b) implies (c) follows by Lemma 3.4 and by the fact that if  $X$  is a normal space,  $x$  is a point in  $X$  and  $X \setminus \{x\}$  is a perfectly normal space, then  $X$  is a perfectly normal space whenever  $x$  is a  $G_\delta$  set in  $X$ .

**THEOREM 3.6.** *Let  $X$  be a locally compact space. Then  $X$  is sequentially locally compact - closed iff  $X$  is countably compact.*

*Proof.* It is obvious that if  $X$  is countably compact then  $X$  is sequentially locally compact - closed. Let  $X$  be a sequentially locally compact - closed space and let us assume that  $X$  is not countably compact. Then there exists a sequence  $\{x_n\}_{n=1}^\infty$  of distinct points of  $X$  without adherent points. Let  $\omega X$  be the Alexandroff compactification of  $X$  (see [En]) and  $y = \omega X \setminus X$ . It is easy to verify that  $\lim_{n \rightarrow \infty} x_n = y$ . Thus  $X$  is not sequentially locally compact - closed. Contradiction.

**THEOREM 3.7.** *For  $\mathcal{D} = \text{paracompact or metric}$  if  $X$  is a  $\mathcal{D}$ -space then the following conditions are equivalent:*

- (a)  $X$  is  $\mathcal{D}$ -closed
- (b)  $X$  is compact
- (c)  $X$  is sequentially  $\mathcal{D}$ -closed
- (d)  $X$  is countably compact.

*Proof.* For the equivalence of (a) and (b) see [SSe] and for the equivalence of (b) and (d) see [En]. Obviously (d) implies (c). The proof that (c) implies (d) for  $\mathcal{D} = \text{paracompact}$  follows by Corollary 3.1 and by the fact that if  $X$  is a regular space,  $x \in X$  and  $X \setminus \{x\}$  is a paracompact space then  $X$  is a paracompact space. For  $\mathcal{D} = \text{metric}$  it follows by Corollary 3.1 and by the fact that if  $X$  is a regular first countable space,  $x \in X$  and  $X \setminus \{x\}$  is a metric space then  $X$  is a metric space.

The spaces satisfying the equivalent conditions (a) - (f) of Theorem 2.1 and the equivalent conditions (a), (b) of Theorem 2.3 and Theorem 2.7 and the equivalent conditions (a) - (c) of Theorem 2.5 are in fact natural generalizations of the countable compactness. Moreover for  $\mathcal{D} = US$  ( $SUS$ ) the sequentially  $\mathcal{D}$ -closed spaces are precisely the sequentially compact (countably compact) spaces [DGo]. The next theorem shows that some properties of the countably compact spaces are valid also for the sequentially  $\mathcal{D}$ -closed spaces.

**THEOREM 3.8.** *Let  $n \in \mathbb{N}$  and  $\mathcal{D}$  be one of the following classes of topological spaces:  $US, SUS, S(n)$ , regular, completely Hausdorff, completely regular, normal, perfectly normal, locally compact, paracompact or metric. Then the following conditions are satisfied:*

- (a) *Sequentially  $\mathcal{D}$ -closedness is preserved by continuous functions onto a  $\mathcal{D}$  space.*
- (b) *If a product of nonvoid spaces is sequentially  $\mathcal{D}$ -closed then each coordinate is sequentially  $\mathcal{D}$ -closed.*
- (c) *Every sequentially  $\mathcal{D}$ -closed space is pseudocompact.*

*Proof.* Obviously (a) implies (b). Clearly (a) and (c) are true when sequentially  $\mathcal{D}$ -closedness coincides with countable compactness or sequential compactness, i.e. for  $\mathcal{D} = US, SUS$ , completely regular, normal, perfectly normal, locally compact, paracompact or metric. For the others (a) follows from Theorem 2.1, Theorem 2.3 and Theorem 2.5. To see that (c) is true let  $f : X \rightarrow \mathbb{R}$  be a continuous function. Then  $f(X)$

is a sequentially  $\mathfrak{S}$ -closed metric space by (a). Thus  $f(X)$  is a compact space by Theorem 2.7 and Theorem 3.7. This shows that  $f$  is bounded.

**THEOREM 3.9.** *Let  $n \in \mathbb{N}$ . The following conditions are valid:*

- (a)  $S(n)(1)$ -closedness is preserved by continuous functions onto an  $S(n)$ -space.
- (b) If a product of nonvoid spaces is an  $S(n)(1)$ -closed space then each coordinate is  $S(n)(1)$ -closed.
- (c) Every  $S(n)(1)$ -closed space is pseudocompact.

*Proof.* (a) follows by Theorem 2.8 and (a) implies (b). We shall prove (c). Let  $f : X \rightarrow \mathbb{R}$  be a continuous function. Then  $\mathcal{U} = \{f^{-1}(-k, k)\}_{k=1}^{\infty}$  is a countable regular cover of  $X$ . Since  $X$  is  $S(n)(1)$ -closed then by Theorem 2.8 we can choose a finite proximate subcover of  $X$ . This implies that  $f$  is bounded.

The above theorem for  $n = 1, 2$  is proved by R. Stephenson [Ste2].

Let  $\mathfrak{S}$  be a class of topological spaces.  $X \in \mathfrak{S}$  is called  $\mathfrak{S}$ -minimal iff  $X$  has no strictly coarser  $\mathfrak{S}$  topologies. (For  $\mathfrak{S}$ -minimal spaces see [BPS]).

**LEMMA 3.10.** *Let  $n \in \mathbb{N}$ ,  $\mathfrak{S} = S(n)$ , regular, completely Hausdorff or completely regular and  $X$  be a  $\mathfrak{S}$ -minimal space. If  $x \in X$  and  $x$  is not a limit point for a non trivial sequence in  $X$  then  $Y = X \setminus \{x\}$  is a sequentially  $\mathfrak{S}$ -closed space.*

*Proof.* Let us assume that  $Y$  is not sequentially  $\mathfrak{S}$ -closed space. Then there exists an open elementary  $\mathfrak{S}$ -filter  $\mathcal{F}_1$  on  $Y$  without adherent points. Let  $\mathcal{F}_x$  be the filter of neighbourhoods of the point  $x$  on  $X$ . We consider the filter  $\mathcal{F} = \{U \mid U = V \cup W, V \in \mathcal{F}_1, W \in \mathcal{F}_x\}$ . Obviously  $\mathcal{F}$  is an open elementary  $\mathfrak{S}$ -filter on  $X$  and  $x$  is the unique adherent point for  $\mathcal{F}$ . Let  $\mathcal{F}'_1$  be a maximal open elementary filter on  $Y$  containing  $\mathcal{F}_1$ . Suppose that  $\{x_k\}_{k=1}^{\infty}$  determines  $\mathcal{F}'_1$ . But  $\lim_{k \rightarrow \infty} x_k \neq x$ . Thus  $\mathcal{F} \subsetneq \mathcal{F}_x$ . But this contradicts to the  $\mathfrak{S}$ -minimality of  $X$ .

**COROLLARY 3.11.** *Let  $X$  be a compact Hausdorff space and  $x \in X$ . The point  $x$  is not the limit of a some (non trivial) sequence of  $X$  iff  $X \setminus \{x\}$  is a countably compact space.*

*Proof.* Every compact Hausdorff space is a minimal completely regular space (see [Ba]). Now the corollary holds by Lemma 3.10 and Theo-

rem 2.7.

Let  $C(X)$  be the set of all real - valued continuous functions on a space  $(X, \mathcal{T})$ . The weak-topology  $\mathcal{T}_\omega$  on  $X$  is the smallest topology on  $X$  such that all functions in  $C(X)$  are continuous. Clearly  $\mathcal{T}_\omega$  is coarser than  $\mathcal{T}$  and the space  $(X, \mathcal{T}_\omega)$  is completely regular iff  $(X, \mathcal{T})$  is completely Hausdorff.

**THEOREM 3.12.** *Let  $(X, \mathcal{T})$  be a completely Hausdorff space.  $(X, \mathcal{T})$  is sequentially completely Hausdorff - closed iff  $(X, \mathcal{T}_\omega)$  is countably compact.*

*Proof.* It follows by Theorem 2.7 and by the fact that  $(X, \mathcal{T})$  is sequentially completely Hausdorff - closed iff  $(X, \mathcal{T}_\omega)$  is sequentially completely regular - closed.

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