Using Geometer's Sketchpad to Construct Pop-up Polyhedra as a Tool for Classroom Study of Geometry

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Abstract

This paper demonstrates the use of Geometer's Sketchpad to construct 3-dimensional pop-up polyhedra for students to handle in the study of Platonic and Archimedean solids. It explores some history of the use of pop-up polyhedra to illustrate mathematical shapes, as well as survey strategies for construction. A cloth book for a new granddaughter Eleanor inspired this project. Its pop-ups were an early favorite of Eleanor and her parents.

Historical Context

Visualizing three-dimensional objects has challenged generations of geometry students. Pop-up models take up little space yet provide rigid figures to study.

Examples prior to 1950. Tufte [1] includes a model of a regular tetrahedron fastened to a page that replicates one from John Dee's preface to Henry Billingsley's *Elements of Geometrie.* Published in London in 1570 this is the first English translation of Euclid's *Elements*. The University of Toronto Library has an original copy of this text and provides pictures on the Internet [2]. Cameo Wood posted a movie on You Tube that demonstrates the operation of these pop-ups [3]. John Lodge Cowley's *Geometry Made Easy* first published in 1752 also used pop-up models to illustrate solid geometry. The University of Michigan's rare books library has a copy believed to date from the 1760's [4]. Steinhaus [5] included a dodecahedron that consisted of two six pentagon "flowers" placed back to back with an elastic band threaded alternately above and below the pentagons on opposite sides (see also Ball [6]). Struyk [7] based three models on basic nets of polyhedra with pyramids glued to each of the faces to illustrate a cube with an inscribed regular tetrahedron, a rhombic dodecahedron with an inscribed cube and a rhombic dodecahedron with inscribed regular octahedron. These are interesting folding models, but they do not fold flat for easy transport.

More recent examples. In 1972 Trigg [8] cut a regular octahedron along some edges to achieve a collapsible polyhedron net augmented with regular triangles in strategic places. He described similar constructions for a collapsible tetrahedron from earlier papers. By 1982 Johnny Ball included some basic pop-up models of Platonic solids [6]. In 1992 Baranowski [9, 10] explored properties of some collapsible models of the tetrahedron, cube, octahedron, rhombic dodecahedron and cubo-octahedron. He formulated conditions that these constructions should obey: "Each vertex of the folded model must belong to parts of at least two faces, each paper fold is along an edge, … [and] the surface of the model should have 'no ends'." His models are assembled like a ring. Hans Walser and Scott Johnson have written extensively about folding polyhedra [11, 12]. In 1996 Walser shared an office with Jean Pederson who was working on a project involving extensions to the Pascal Triangle. Pederson wanted a model that would fold flat in a brief case to travel to lectures [11]. Walser developed a model that interested many of Pederson's students. One, Scott Johnson, was eager to help in the development of a number of pop-up polyhedra. Johnson and Walser described their models and the techniques they developed to build them in [11]. Most of the models they describe are models of the Platonic solids although Walser built a cuboctahedron [13] as the original motivation for their project.

A Catalog of Techniques

Definition of Terms. Ball [6] used rubber bands for automated opening of his polyhedra. Johnson and Walser use rubber bands, pull strings, and paper fastener rotation centers [11]. Dee [2] and Cowley [4] simply used flaps attached to a page and fingers to lift and erect the models. Johnson and Walser [11] describe "edge-jumping" models that open as the result of two edges being caused to move towards each other. The cube, octahedron, icosahedron and dodecahedron all have parallel edges that make this possible. Since the tetrahedron has opposite edges that are not parallel (but are perpendicular when collapsed) they had to design "webbed hinges" that incorporate folding triangles at vertices of adjacent faces to include triangles that touch only at a vertex. They built a "face-jumping" icosahedron and a "vertex-jumping" cube that use brass paper fasteners as pivots. They also introduce "spiral models" where the top and bottom layers turn relative to each other as the models collapse.

Using Geometer's Sketchpad to Construct the Models

Tools from Sketchpad. Geometer's Sketchpad [13] (similar to Cabri) is a tool that can provide precise constructions of the polygons and fold lines. In the process students learn to identify centers and angles of rotation, length and directions of translations and placement of lines of reflection. Ball, and Johnson and Walser [11] provided the examples for classroom use.

A simple place to start. Although Ball [6] used the end of an envelope for his construction of a regular tetrahedron, we will use the tools of Geometer's Sketchpad [13]. A regular tetrahedron has 4 equilateral triangles as faces. We need to fold two of these triangles along their altitudes to enable the model to collapse. Use a unit segment that is not part of the model (see AB in Figure 1 left) to begin construction. This permits us to adjust the size of the model without distortion at some time in the future. First construct an equilateral triangle using this unit. Construct a line through two points C and D. Use C as center and AB as radius to construct a circle. Locate one of its intersections with CD. Label it E. Now construct a circle with radius AB and center E. Locate both intersections of our two circles (F and G). Note CF=FE=CE and CE=EG=CG (Figure 1 left). Connect pairs of these points with segments to construct two equilateral triangles with side lengths equal to AB. Hide any construction objects we no longer need. We need two more faces that will be folded. Construct four more triangles on the sides of our original triangles. Half of each of these triangles will be parts of the faces and half will be tabs to glue together to facilitate the pop-up. Mark segment FE as a mirror and reflect triangle CEF over EF using the reflection transformation. Similarly reflect triangle CEF over CF, and triangle CFG over CG and GE (Figure 1 center). Re-label the images of C and E as shown (C_1 and C_2 and E_1 and E_2). Select point C and segment E_1F to construct a line perpendicular to E_1F through C. This line is also perpendicular to C_2G. Why?Complete the model by making all edges of the final model thick and coloring the tabs. Cut around the perimeter of the figure and fold all interior segments away from you (Figure 1 right). With glue on the yellow tabs, place triangle C_1JE on top of C_2KE and E_1HC on top of E_2IC. Push gently on the sides and a tetrahedron will pop-up.

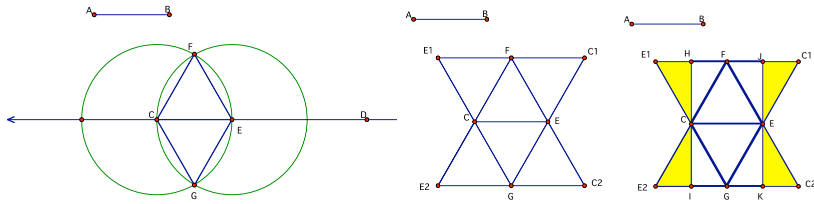

Figure 1: *Geometer's Sketchpad construction of the tetrahedron.*

The cube (Figure 2 left) and octahedron (Figure 2 right) use Johnny Ball's model from [6] page 134 in a similar way.

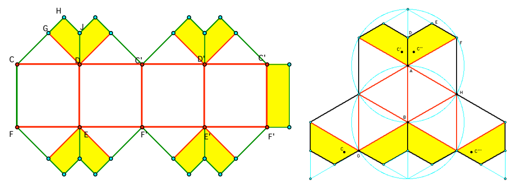

Figure 2: *Sketchpad construction of a cube (left) and an octahedron (right).*

An icosahedron with pivots. This model uses Johnson and Walser's face jumping icosahedron to illustrate the use of pivots. Central (purple shaded) equilateral triangles are connected with trapezoids made of equilateral triangles attached to triangular tabs that pivot. See Figure 3. Cut and fold as before. With the tabs folded to the back, arrange two copies with the central triangles back to back and oriented so that a base of one is opposite the base of the other. Matching one holed tab from the back with a neighboring holed tab from the front insert a brass paper fastener out of sight at 6 different places. Gentle pressure on the "star points" produces an icosahedron. See Figure 3. This pivot method also produces an interesting cube that shows the intersection of the plane intersecting the perpendicular bisector of the diagonal of a cube with the cube. When assembled the folds in the faces of the cube create a regular hexagon.

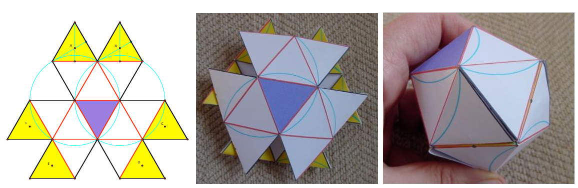

Figure 3: *An icosahedron with pivots constructed, assembled, and activated.*

Original Pop-up Figures

A Truncated Octahedron and Rhombicuboctahedron. A natural extension of the Platonic models is to see what Archimedean solids might be modeled this way.

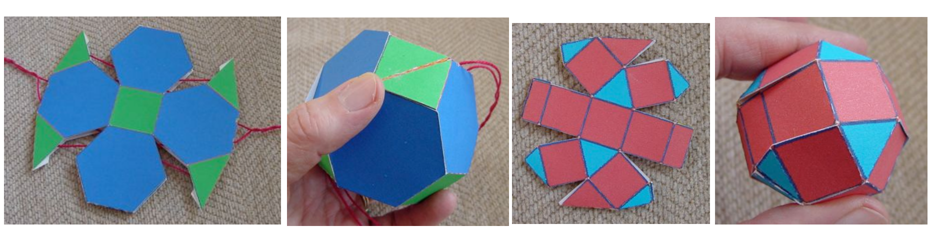

Figure 4: *The truncated octahedron (left) and rhombicuboctahedron (right) flattened and popped up.*

The truncated octahedron and rhombicuboctahedron are two of the simple Archimedean solids. Decide which polygon to use as you start the Geometer's Sketchpad construction. For both models use the parallel square faces as the starting place. Any folded polygons will be folded symmetrically along altitudes or diagonals. See Figure 4.

Conclusion

Ideas for further study. Magnus J. Wenninger [14,15] describes the remaining 11 Archimedean solids and stellated and compound polyhedra.

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