The Weil-Petersson Hessian of Length

Outline: I. A few remarks on Riemannian / Harmonic Maps Approach to Teichmüller Theory

II. An example of computing with this perspective: the WP Hessian of length.

I. ① Background on Harmonic maps between surfaces

a) ENERGY

S a differentiable surface of genus $g \geq 2$. $(S, \sigma(z)|dz|^2)$, $(S, \rho(w)|dw|^2)$, Riemannian surfaces. For a map $w: (S, \sigma(z)|dz|^2) \to (S, \rho(w)|dw|^2)$, define the energy $E(w: \sigma \to \rho)$ of the map w to be

$$E(w : \sigma \to \rho) = \frac{1}{2} \iint_S \left\{ \|w_* e_1\|_\rho^2 + \|w_* e_2\|_\rho^2 \right\} d\,\mathrm{Area}(\sigma)$$

where $\{e_1, e_2\}$ is an orthonormal frame for σ

$$= \iint_S \left\{ \frac{\rho(w(z))}{\sigma(z)} |w_z|^2 + \frac{\rho(w(z))}{\sigma(z)} |w_{\bar z}|^2 \right\} \sigma(z)\, dz d\bar z$$

Note that the $\sigma(z)$'s cancel, so

$$E(w : \sigma \to \rho) = \iint_S \left\{ \rho(w(z)) |w_z|^2 + \rho(w(z)) |w_{\bar z}|^2 \right\} dz \wedge d\bar z$$

and so depends on (i) the map w, (ii) the metric ρ on the target, but (iii) only on the conformal structure z of the domain, not the metric $\sigma(z)$.

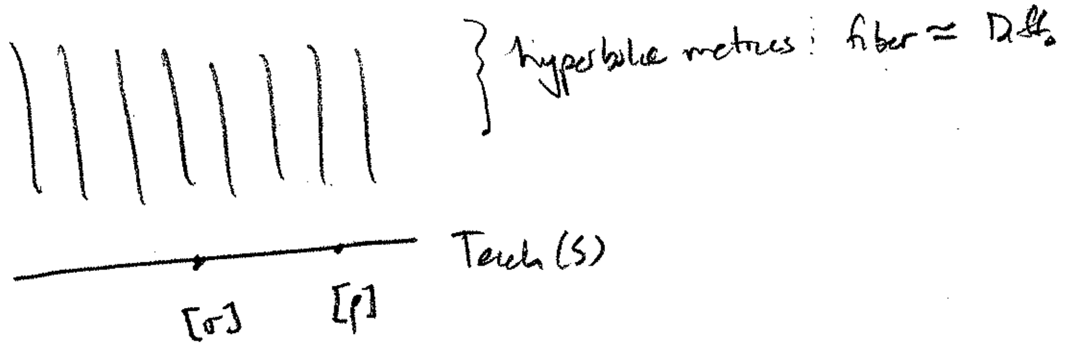
Fix $\sigma(z)$, $\rho(w)$ on S.

<u>Defn</u> A critical point $w: (S, \sigma(z)) \to (S, \rho|dw|^2)$ is called a harmonic map.

Note: A geodesic $\gamma: S^1 \to (S, \rho)$ is a harmonic map of $S^1 \to (S, \rho)$.
Our version above is a generalization of this concept to the setting of maps between surfaces.

b) <u>Foundational Result</u>: (Eells-Sampson '64; D'Alber, Hartman '66; Schoen-Yau, Sampson '78) Let $F: S \to S$ be a diffeomorphism of S to itself. Let $\rho|dw|^2$ be negatively curved. Then there exists a unique energy-minimizing map $w(\sigma, \rho): (S, \sigma) \to (S, \rho)$ homotopic to F and w is a diffeomorphism (emphasize dw has rank $= 2$).

Who cares?
Well, $\mathrm{Teich}(S) = \{\text{hyperbolic metrics on } S\} / \mathrm{Diff}_0$ is a quotient space, so this result says that, once we fix a basepoint σ, we can find a well-defined representative of each element of Teichmüller space in the space of hyperbolic metrics



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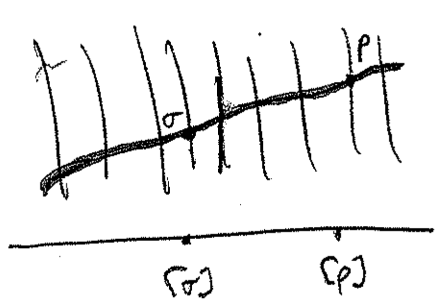


Metaphor: If (S,σ) is a stone surface and ρ is a differently-shaped rubber metric we are trying to fit on (S,σ), the harmonic map is the fitting which requires the least energy among all fittings.

c) HOPF DIFFERENTIALS

Two perspectives:

(i) The function $E(\rho) = E(\underbrace{w(\sigma,\rho)}_{\text{harmonic map}})$ defines a function

$$E : \mathrm{Teich}(S) \longrightarrow \mathbb{R}$$
$$\rho \longmapsto E(w(\sigma,\rho) : \sigma \to \rho)$$

Then $dE\big|_\sigma$ defines an element of $T^*_\sigma \mathrm{Teich}(S)$.

Hence $dE\big|_\sigma \in T^*_\sigma \mathrm{Teich}(S) \cong QD(\sigma)$; ie something in this situation is holomorphic.

Pull $\rho(w)|dw|^2$ back to (S,z) by the harmonic map $w = w(\sigma,\rho)$. That metric decomposes by type as

$$\rho(w)|dw|^2 = \rho w_z \bar{w}_z \, dz^2 + \sigma \frac{\rho(w)}{\sigma(z)} \{ |w_z|^2 + |w_{\bar z}|^2 \} dz d\bar z + \overline{\rho w_z \bar{w}_z \, dz^2}$$

$$= \Phi \, dz^2 + \sigma \, e(w : \sigma \to z) \, dz d\bar z + \bar\Phi \, d\bar z^2$$

Now w harmonic $\Longleftrightarrow$ Φ holomorphic quadratic differential.

(using w a diffeomorphism)

(ii) Other perspective: Euler-Lagrange equation 2nd order. In some settings (2-d qualifies), expect a conservation law, ie a reduction of this problem to a 1st order equation, ie. $\partial_{\bar z} \phi = 0$.

This yields a map

$$\Phi : \mathrm{Teich}(S) \longrightarrow QD(\sigma)$$

$$\rho \longmapsto w(\sigma,\rho) \mapsto w(\sigma,\rho)^*\rho|dw|^2 \mapsto \left(w(\sigma,\rho)^*\rho\right)^{2,0} = \Phi(\rho)$$

Thm (a) (Sampson, W) Φ is a (surjective) diffeomorphism

(b) $\Phi^{-1} : QD(\sigma) : \mathrm{Teich}(S)$ extends to a homeomorphism

$\overline{\Phi^{-1}} : QD \cup PQD \to \overline{\mathrm{Teich}(S)}^{\text{Thurston}}$

HARMONIC MAP COORDINATES FOR TEICHMULLER SPACE

Pick $\phi \in QD(\sigma)$, look at $t\phi \xrightarrow[\Phi^{-1}]{}$ a family ρ_t of hyperbolic metrics

Then by construction

$$\rho_t = t\phi\, dz^2 + \sigma\Big\{\underbrace{\frac{\rho(w)}{\sigma(z)}|w_z|^2}_{\mathcal{H}} + \frac{\rho(w)}{\sigma(z)}|w_{\bar z}|^2\Big\}dz d\bar z + \overline{t\phi\, dz^2}$$

$$= t\phi\, dz^2 + \sigma\Big\{\mathcal{H} + \frac{t^2|\phi|^2}{\sigma^2}\cdot\frac{1}{\mathcal{H}}\Big\}dz d\bar z \quad + t\bar\phi\, d\bar z^2$$

How does $\mathcal{H}$ relate to ϕ? (After all, ρ_t depends only on t, ϕ)

well $K(\rho_t) \equiv -1 \iff \Delta_\sigma \log \mathcal{H} = 2\mathcal{H} - 2\frac{t^2|\phi|^2}{\sigma^2}\cdot\frac{1}{\mathcal{H}} - 2 \quad (*)$

Study this equation!

Note: (i) For $t=0$, we have $\mathcal{H}|_{t=0} \equiv 1$

(ii) Take $\frac{d}{dt}\Big|_{t=0}$ of $*$ to get $\Delta \frac{\dot{\mathcal{H}}}{\mathcal{H}} = 2\dot{\mathcal{H}}$ i.e. $\Delta\dot{\mathcal{H}} = 2\dot{\mathcal{H}}$

By maximum principle, have $\dot{\mathcal{H}} \equiv 0$.

(iii) $\Delta \ddot{\mathcal{H}} = 2\ddot{\mathcal{H}} - \frac{4|\phi|^2}{\sigma^2}$

$\Rightarrow \ddot{\mathcal{H}} = -2(\Delta-2)^{-1}\frac{2|\phi|^2}{\sigma^2}$

(iv) etc to all orders: (Find, e.g. $\frac{d^2}{dt^2}E(\sigma,\rho_t) = 2\int \frac{\|\phi\|^2}{\sigma^2}\, dA = 2\langle\phi,\phi\rangle_{WP}$; $\frac{d^4}{dt^4}E(\sigma,\rho_t) = 6\int |\phi|^2(-2(\Delta-2)^{-1}\frac{|\phi|^2}{\sigma^2})$

Conclude; in this section of Diffs,

$$\rho_t = t\phi\, dz^2 + \left(1 + t^2\left\{\frac{|\phi|^2}{\sigma^2} + -2(\Delta-2)^{-1}\frac{|\phi|^2}{\sigma^2}\right\} + O(t^4)\right) dz d\bar{z} + t\bar{\phi}d\bar{z}^2$$

Corollary: The path ρ_t is WP-geodesic at $t=0$.

[(Jost): Higher orders allow for computation of R_{WP}.]

II A computation

Thm Let $\Gamma(t)$ be a WP geodesic in Teich(S), with $\dot{\Gamma}(0) = \frac{\bar{\phi}}{\sigma}$. [Here $\Gamma(t) = [\rho_t] = ds_{\rho_t}$.] Let $[\gamma]$ denote the free homotopy class of a simple closed curve, represented by the ds_{ρ_t} (hyperbolic) geodesic $\gamma_t = \gamma_t(s)$ ← s = arclength parameter along γ

t = time parameter along WP geodesic $\Gamma(t)$

Let $\ell_t = \ell_{\Gamma(t)}([\gamma]) = \ell_{\rho_t}(\gamma_t)$

Then

$$\frac{d^2}{dt^2}\ell(t) = \int_{\gamma_0} -2(\Delta-2)^{-1}\frac{|\phi|^2}{\sigma^2}\, ds$$

$$+ \frac{1}{2\sinh(\ell/2)}\iint_{\gamma_0\times\gamma_0}\left[\operatorname{Im}\frac{\bar{\phi}}{\sigma}(p)\right]\left[\operatorname{Im}\frac{\bar{\phi}}{\sigma}(q)\right]\left\{\cosh\left(d(p,q) - \ell/2\right)\right\} ds(p)\, ds(q)$$

So only need to compute 2nd term $D_{22}^2\, \ell(\rho_0,\gamma_0)[\mathring{\gamma},\mathring{\gamma}]$.

Let us call this variational field $\mathring{\gamma}$ by V to suggest that it's a vector field along the curve.

By elementary Riemannian geometry we know that

$$D_{22}^2\, \ell(\rho_0,\gamma_0) = \frac{d^2}{dt^2}\ell(\rho_0,\gamma_t) = \int_{\gamma_0} |\nabla V|^2 + |V|^2\, ds$$

$$= -\int V(V''-V)\, ds$$

Here's the main idea: since γ_t is a geodesic on ρ_t, the ρ_t-geodesic curvature of γ_t must vanish, i.e. $\kappa(\rho_t,\gamma_t)=0$. Take $\frac{d}{dt}$ of this eq

$$\text{So}\quad \frac{d}{dt}\kappa(\rho_0,\gamma_t) = -\frac{d}{dt}\kappa(\rho_t,\gamma_0)$$

classical $\swarrow$ $\qquad$ $\searrow$ by our explicit form of ρ_t

$$= V''-V \qquad\qquad = -\frac{\partial}{\partial y}\left\{\frac{\mathrm{Im}\,\phi}{\sigma}\right\}$$

Conclude $\quad V''-V = -\frac{\partial}{\partial y}\left\{\frac{\mathrm{Im}\,\phi}{\sigma}\right\} \equiv -\tilde{f}_y$

Useful to consider a primitive $U(y)=\int_{y_0}^{y} V(s)\,ds$ so that U satisfies $\quad U''-U = -\tilde{f}$

$$\text{Then}\quad \frac{d^2}{dt^2}\ell(\rho_0,\gamma_t) = -\int V(V''-V) \qquad \text{) substitutions}$$

$$= \int U'\tilde{f}' \qquad \text{) parts}$$

$$= -\int U''\tilde{f} \qquad \text{) substitutions, parts}$$

$$= -\int \tilde{f}^2 - \int (U')^2 + U^2$$

$$\frac{d^2}{dt^2}\,\ell(t) = \int_{\gamma_0} -2(\Delta-2)^{-1}\frac{|\phi|^2}{\sigma^2}\,ds$$

$$+ \frac{1}{2\sinh \ell/2}\iint_{\gamma_0\times\gamma_0}\left[\mathrm{Im}\,\frac{\Phi}{\sigma}(p)\right]\left[\mathrm{Im}\,\frac{\Phi}{\sigma}(q)\right]\left\{\cosh\left(d(p,q)-\ell/2\right)\right\}ds(p)\,ds(q)$$

Notes: 1) a) 1^{st} term >0, 2^{nd} term ≥ 0 (Note kernel $\dfrac{\cosh(d(\cdot,\cdot)-\ell/2)}{2\sinh \ell/2} \geq 0$, or write 2^{nd} term as an "energy".)

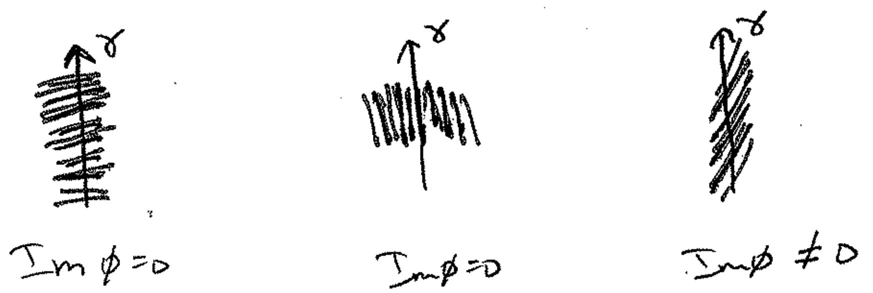
b) Here $\mathrm{Im}\,\frac{\Phi}{\sigma}$ makes sense, once I tell you that I'm secretly using Fermi coordinates along the curve

[c) Not an infinite series!]

2) First term is restriction of global term $-2(\Delta-2)^{-1}\frac{|\phi|^2}{\sigma^2}$ (measuring total deformation on surface).

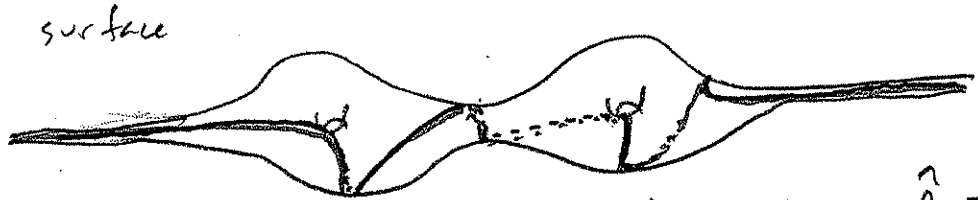
Second term concerns how $\gamma(t)$ gets "sheared" by $\dot{\Gamma}$

Foliation of ϕ



3) a) Formula extends to measured laminations

b) Consider a proper geodesic γ "connecting punctures" on a punctured surface



$\ell(\gamma) = \infty$ but can be "regularized" to a function $\hat{\ell}$: Teich$(S) \to \mathbb{R}$ depending only on a choice of constants. Then $\frac{d^2}{dt^2}\hat{\ell} < \infty$ and is given by a version of the formula.

4) Take limits and integrate directly to get a concise proof of Wolpert's result that Thurston metric = $\frac{4}{3}$ WP.

Proof of formula:

Note $\ell(t) = \ell(\rho_t, \gamma_t)$ is a function of two variables: the metric and the curve

variable #1 — variable #2

Because γ_t is a ρ_t-geodesic, we know that it's critical for length: $D_2 \ell(\rho_t, \gamma_t)[\dot\gamma] = 0$

[Warmup (Gardiner's Formula): $\frac{d}{dt}\ell(t) = D_1 \ell(\rho_0, \gamma_0)[\dot\rho] + \cancel{D_2 \ell(\rho_0, \gamma_0)[\dot\gamma]}^{\,0}$

$$= \frac{d}{dt}\int_{\gamma_0} \sqrt{\rho_t}$$

$\rho_t = 2t\,\mathrm{Re}\,\phi + O(t^2)$

$$= \int_{\gamma_0} \frac{2\,\mathrm{Re}\,\phi}{\sigma}\, ds \,.\ ]$$

2nd Derivative: $$\frac{d^2}{dt^2}\ell(t) = D_1^2 \ell[\dot\rho, \dot\rho] + 2 D_1 D_2 \ell[\dot\rho, \dot\gamma] + D_2^2 \ell(\dot\gamma, \dot\gamma) \qquad (A)$$

a) First, eliminate the cross term: because $D_2 \ell(\rho_t, \gamma_t)[\dot\gamma] = 0$,

$$0 = \frac{d}{dt} D_2 \ell(\rho_t, \gamma_t)[\dot\gamma] = D_1 D_2 \ell[\dot\rho, \dot\gamma] + D_2^2 \ell[\dot\gamma, \dot\gamma]. \qquad (B)$$

Substitute (B) into (A) to find

$$\frac{d^2}{dt^2}\ell(t) = D_1^2 \ell[\dot\rho, \dot\rho] - D_2^2 \ell[\dot\gamma, \dot\gamma]$$

But, first term is easy, as we know $\rho_t = \sigma|dz|^2 + 2t\{\mathrm{Re}\,\phi\, dz^2\} + t^2\{\text{explicit stuff}\} + O(t^4)$

Last piece:

Our answer is now in terms of $\frac{|\phi|^2}{\sigma}$, $\left(\frac{\mathrm{Re}\,\phi}{\sigma}\right)^2$, $\left(\frac{\mathrm{Im}\,\phi}{\sigma}\right)^2$ and U.

But for U satisfying $U'' - U = -J \quad \left(= -\frac{\mathrm{Im}\,\phi}{\sigma}\right)$,

we can write a solution U as

$$U(s) = -\int J(t)\, K(s,t)\, dt$$

where $K(s,t)$ satisfies

$$\left(\frac{d^2}{dt^2} - 1\right) K(s,t) = \delta_s(t)$$

So is some combination of $\cosh(t)$ and $\sinh(t)$.

The rest is assembly.